Problem 6.2
Consider the charge and current densities for a single point charge \( q \). Formally, these are given by

\[
\rho(x', t') = q \delta [x' - r(t')], \quad J(x', t') = q v(t') \delta [x' - r(t')],
\]

where \( r(t') \) is the charge’s position at time \( t' \) and \( v(t') \) is its velocity. While evaluating the expressions involving retarded time, we must put \( t' = t_{\text{ret}} = t - R(t')/c \), where \( R = x - r(t') \) and \( R = x - x'(t') \) inside delta functions.

a) We are to show that

\[
\int d^3 x' \delta [x' - r(t_{\text{ret}})] = \frac{1}{\kappa},
\]

where \( \kappa \equiv 1 - \mathbf{v} \cdot \dot{\mathbf{R}}/c \), where \( \kappa \) is evaluated at retarded time.

Recall the trivial identity of the Dirac \( \delta \)-function,

\[
\delta [f(x)] = \left. \frac{\delta(x - x_0)}{\partial f/\partial x} \right|_{x=x_0}
\]

where \( x_0 \) is a root of \( f(x) \). Setting \( f(x') = [x' - r(t_{\text{ret}})] \) in the expression above, \( x_0 \) is such that \( x_0 = r(t_{\text{ret}}) \). Inserting this directly, we find

\[
\delta [x' - r(t_{\text{ret}})] = \delta(x' - x_0) \left( \frac{\partial}{\partial x'} (x' - r(t_{\text{ret}})) \right)^{-1}_{x'=x_0},
\]

\[
= \delta(x' - x_0) \left( 1 - \frac{\partial}{\partial x'} r(t_{\text{ret}}) \right)^{-1}_{x'=x_0},
\]

\[
= \delta(x' - x_0) \left( 1 - \frac{\partial r}{\partial t} c |x' - x| \right)^{-1}_{x'=x_0},
\]

\[
= \delta(x' - x_0) \left( 1 - \frac{\partial r}{\partial t} c |x - x'| \right)^{-1}_{x'=x_0},
\]

\[
= \delta(x' - x_0) \left( 1 - \mathbf{v} \cdot \dot{\mathbf{R}} \right)^{-1}_{c},
\]

\[
= \frac{\delta(x' - x_0)}{\kappa},
\]

\[
\therefore \int d^3 x' \delta [x' - r(t_{\text{ret}})] = \frac{1}{\kappa}.
\]

b) Starting with the Jefimenko generalizations of the Coulomb and Biot-Savart laws, we are to use the expressions for the charge and current densities for a point charge and the result of part a above to obtain the Heaviside-Feynman expressions for the electric and magnetic fields of a point charge.

Let us begin with the electric field. The Jefimenko generalization is given by

\[
E(x, t) = \frac{1}{4 \pi \epsilon_0} \int d^3 x' \left\{ \frac{\dot{R}}{R^2} [\rho(x', t')]_{\text{ret}} + \frac{\dot{R}}{c R} \left[ \frac{\partial \rho(x', t')}{\partial t'} \right]_{\text{ret}} - \frac{1}{c^2 R} \left[ \frac{\partial J(x', t')}{\partial t'} \right]_{\text{ret}} \right\}.
\]
Directly inserting our current and charge densities, following out the algebra and using our result from part a, we have

\[
E(x, t) = \frac{1}{4\pi \epsilon_0} \int d^3x' \left\{ \frac{\hat{R}}{R^2} \left[ \rho(x', t') \right]_{\text{ret}} + \frac{\hat{R}}{cR} \left[ \frac{\partial \rho(x', t')}{\partial t'} \right]_{\text{ret}} - \frac{1}{c^2 R} \left[ \frac{\partial J(x', t')}{\partial t'} \right]_{\text{ret}} \right\},
\]

\[
E(x, t) = \frac{1}{4\pi \epsilon_0} \int d^3x' \left\{ \frac{\hat{R}}{R^2} q \left[ \delta(x' - r(t')) \right]_{\text{ret}} + \frac{\hat{R}}{cR} q \left[ \frac{\partial}{\partial t'} \delta(x' - r(t')) \right]_{\text{ret}} - \frac{1}{c^2 R} q \left[ \frac{\partial}{\partial t'} v(t') \delta(x' - r(t')) \right]_{\text{ret}} \right\},
\]

\[
E(x, t) = \frac{q}{4\pi \epsilon_0} \int d^3x' \left\{ \frac{\hat{R}}{R^2} \left[ \delta(x' - r(t')) \right]_{\text{ret}} + \frac{\hat{R}}{cR} \left[ \frac{\partial}{\partial t} \delta(x' - r(t')) \right]_{\text{ret}} - \frac{1}{c^2 R} \left[ \frac{\partial}{\partial t} v(t') \delta(x' - r(t')) \right]_{\text{ret}} \right\},
\]

\[
\therefore E(x, t) = \frac{q}{4\pi \epsilon_0} \left\{ \frac{\hat{R}}{R^2} \left[ \frac{\partial}{\partial t} \delta(x' - r(t')) \right]_{\text{ret}} - \frac{\partial}{\partial t} \left[ \frac{1}{c^2 R} v(t') \delta(x' - r(t')) \right]_{\text{ret}} \right\}.
\]

Notice that in the above, we have used the fact that \( R \) does not explicitly depend on \( t \).

Let us now do the analogous calculation for the magnetic field. The Jefimenko generalization is given by

\[
B(x, t) = \frac{\mu_0}{4\pi} \int d^3x' \left\{ [J(x', t')]_{\text{ret}} \times \frac{\hat{R}}{R^2} + \left[ \frac{\partial J(x', t')}{\partial t'} \right]_{\text{ret}} \times \frac{\hat{R}}{cR} \right\}.
\]

Directly inserting our current and charge densities and computing directly, we have

\[
B(x, t) = \frac{\mu_0 q}{4\pi} \int d^3x' \left\{ [v(x', t')]_{\text{ret}} \times \frac{\hat{R}}{R^2} + \left[ \frac{\partial v(x', t')}{\partial t'} \right]_{\text{ret}} \times \frac{\hat{R}}{cR} \right\},
\]

\[
B(x, t) = \frac{q}{4\pi} \int d^3x' \left\{ \left[ \frac{\hat{R}}{R^2} \delta(x' - r(t')) \right]_{\text{ret}} \times \frac{\hat{R}}{R^2} + \left[ \frac{\partial}{\partial t'} v(t') \delta(x' - r(t')) \right]_{\text{ret}} \times \frac{\hat{R}}{cR} \right\},
\]

\[
B(x, t) = \frac{q}{4\pi} \left\{ \left[ \frac{\hat{R}}{R^2} \delta(x' - r(t')) \right]_{\text{ret}} + \frac{\partial}{\partial t} \left[ \frac{\hat{R}}{cR} \delta(x' - r(t')) \right]_{\text{ret}} \right\},
\]

\[
\therefore B(x, t) = \frac{\mu_0 q}{4\pi} \left\{ \left[ \frac{\hat{R}}{R^2} \delta(x' - r(t')) \right]_{\text{ret}} + \frac{\partial}{\partial t} \left[ \frac{\hat{R}}{cR} \delta(x' - r(t')) \right]_{\text{ret}} \right\}.
\]