Problem 5.3
Let us consider a right-circular solenoid of finite length \( L \) and radius \( a \) with \( N \) turns per unit length, each carrying current \( I \). In the continuum limit of the current densities on the surface of the solenoid, we are to show that the magnetic induction on the cylindrical axis is

\[
B_z = \frac{\mu_0 NI}{2} \left( \cos \theta_1 + \cos \theta_2 \right),
\]

where the angles are given in a clear diagram not included here.

We can solve this problem rather directly by using our result from problem (5.1) above. There, we showed that the magnetic induction from a closed current is given in terms of the gradient of the solid angle subtended by the circular current (with appropriate conventions for signs). Because of the cylindrical symmetry in the problem at hand, we see that at a particular \( z \),

\[
\Omega = 2\pi \int_{\cos \theta}^{1} \cos \theta d\cos \theta = 2\pi (1 - \cos \theta).
\]

Notice that \( z = a \cos \theta \) and so \( \Omega \) is only a function of \( z \). Hence without loss of generality,

\[
\nabla \Omega = \hat{z} \frac{\partial \Omega}{\partial z}.
\]

Because of the linearity in the induction, the total field \( B_z \) will be given in terms of an integral over the individual currents over \( z \). Therefore,

\[
B_z(z) = \int_{0}^{L} dz' \frac{\mu_0 NI}{4\pi} \frac{\partial \Omega}{\partial z'} \hat{z}',
\]

\[
= \frac{\mu_0 NI}{4\pi} (\Omega(L) - \Omega(0)),
\]

\[
= \frac{\mu_0 NI}{2} (1 - \cos(\pi - \theta_2) - 1 + \cos \theta_1),
\]

\[
\therefore B_z = \frac{\mu_0 NI}{2} (\cos \theta_1 + \cos \theta_2).
\]

Problem 5.5 (c)
We are to derive approximations for the \( z \)- and \( \rho \)-components of the magnetic induction near \( \rho = 0 \) and \( \cos \theta_2 \to 0 \).

Being at one of the ends of the solenoid is equivalent to having \( \cos \theta \) vanish for one of the two angles defined for problem (5.3). This reduces the induction to

\[
B_z = \frac{\mu_0 NI}{2} \cos \theta_1,
\]

where \( \cos \theta_2 = 0 \).

If the solenoid were long, then \( \cos \theta_1 \) can be well-approximated by its Taylor expansion.

\[
\therefore B_z = \frac{\mu_0 NI}{2} \left( 1 - O(\theta^2) \right) \simeq \frac{\mu_0 NI}{2}.
\]

We will have a similar expression estimating the \( \rho \)-component. To obtain this expression, we recall that \( B_z = \frac{\mu_0 NI}{2} \cos \theta_1 \). Because \( z = a \cos \theta_1 \) in our coordinates, this implies that \( B_z = \frac{\mu_0 NI}{2a} \hat{z} \). Therefore,

\[
\frac{\partial B_z}{\partial z} = \frac{\mu_0 NI}{2a}.
\]

Now, by the vanishing divergence of the magnetic induction and the independence of \( B \) on \( \phi \), we see that

\[
\nabla \cdot B = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\rho) + \frac{\partial B_z}{\partial z} = 0.
\]

Using our work above, this implies that

\[
\frac{\partial}{\partial \rho} (\rho B_\rho) = \pm \frac{\mu_0 NI}{2a} \rho.
\]