Specifically, these correspond to the system,

$$
\mu_0 \frac{1d}{2\pi a^2} \sin \phi + \mu_0 \sum_{n=1}^{\infty} n\alpha_n a^{n-1} \sin(n\phi) = \mu \sum_{n=1}^{\infty} \left(n\beta_n a^{n-1} - n\gamma_n a^{-n-1}\right) \sin(n\phi);
$$

$$
-\frac{1d}{2\pi a^2} \cos \phi + \sum_{n=1}^{\infty} n\alpha_n a^{n-1} \cos(n\phi) = \sum_{n=1}^{\infty} \left(n\beta_n a^{n-1} + n\gamma_n a^{-n-1}\right) \cos(n\phi);
$$

$$
\mu \sum_{n=1}^{\infty} (n\beta_n b^{n-1} - n\gamma_n b^{-n-1}) \sin(n\phi) = -\mu_0 \sum_{n=1}^{\infty} n\eta_n b^{-n-1} \sin(n\phi);
$$

$$
\mu \sum_{n=1}^{\infty} (n\beta_n b^{n-1} + n\gamma_n b^{-n-1}) \cos(n\phi) = \sum_{n=1}^{\infty} n\eta_n b^{-n-1} \cos(n\phi).
$$

From the first equation, it is obvious that $\gamma_n = 0 \forall n > 1$ because the polynomials on the right and left hand sides must match order by order. Also from the first equation, we see that $\alpha_n = \mu_r \beta_n$ where $\mu_r = \mu/\mu_0$ for the same reason for $n > 1$. Similarly, by looking at the third equation, we see that $\beta_n = 0 \forall n > 1$; also from the third equation, we have that $\eta_n = \mu_r \gamma_n$ for the same reason for $n > 1$. Putting this together, we see that all of the coefficients vanish for $n \geq 2$.

Let us consider the $n = 1$ coefficients. In this case, we obtain the system (after getting rid of the redundant trigonometric factors),

$$
\begin{bmatrix}
1 & -\mu_r & \mu_r/a^2 & 0 \\
1 & -1 & -1/a^2 & 0 \\
0 & \mu_r & -\mu_r/b^2 & 1/b^2 \\
0 & 1 & 1/b^2 & -1/b^2
\end{bmatrix}
\begin{bmatrix}
\alpha_1 \\
\beta_1 \\
\gamma_1 \\
\eta_1
\end{bmatrix}
= 
\begin{bmatrix}
\frac{-1d}{2\pi a^2} \\
\frac{1d}{2\pi a^2} \\
\frac{1d}{2\pi a^2} \\
0
\end{bmatrix}.
$$

Using a computer algebra package to save time for sleeping, we find immediately that

$$
\eta_1 = \frac{1d}{2\pi a^2} \frac{4\mu_r b^2}{(\mu_r + 1)^2 b^2 - (\mu_r - 1)^2 a^2}.
$$

Therefore, we see that the field outside of the steel cylinder is a two-dimensional dipole field, as in part a, but with a strength reduced by the factor

$$
f = \frac{4\mu_r b^2}{(\mu_r + 1)^2 b^2 - (\mu_r - 1)^2 a^2}.
$$

**Problem 5.19**

Let us consider a magnetically ‘hard’ material in the shape of a right circular cylinder of length $L$ and radius $a$. The cylinder has a permanent magnetization $M_0$, uniform throughout its volume and parallel to its axis.

We are to determine the magnetic field $\mathbf{H}$ and the magnetic induction $\mathbf{B}$ at all points on the axis of the cylinder, both inside and outside.

Let us begin our analysis with Jackson’s equation (5.100). Specifically, we can describe this situation in terms of a magnetic scalar potential, $\varphi_M$ which is given by the expression,

$$
\varphi_M = -\frac{1}{4\pi} \int_{\Omega} \nabla' \cdot \mathbf{M}(x') \frac{d^3 x'}{|x - x'|} + \frac{1}{4\pi} \int_{\partial \Omega} \mathbf{n'} \cdot \mathbf{M}(x') \frac{da'}{|x - x'|}.
$$

Notice that the first integral will vanish because $\mathbf{M}(x')$ is constant throughout the cylinder and hence its divergence vanishes. The second integral will only obtain contributions from the two circular faces of the cylinder because everywhere else the magnetization is orthogonal to the cylinder’s normal. Therefore $\varphi_M = \varphi_1 + \varphi_2$ where $\varphi_1, \varphi_2$ are the contributions from the top ($z > 0$) and bottom faces, respectively.
We can directly compute the contribution of the top face to the scalar potential. Here, we have that

\[ \varphi_1(z) = \frac{1}{4\pi} \int \frac{n' \cdot M(x')}{|x - x'|} \, da', \]

\[ = \frac{1}{4\pi} M_0 2\pi \int_0^a \rho \, d\rho \left( \frac{\rho}{\sqrt{\rho^2 + (z - L/2)^2}} \right)^{1/2}, \]

\[ = \frac{M_0}{4} \int_{(z - L/2)^2}^{a^2 + (z - L/2)^2} \frac{du}{\sqrt{u}}, \]

\[ = \frac{M_0}{2} \left( \sqrt{a^2 + (z - L/2)^2} - |z - L/2| \right). \]

From the obvious symmetry of the problem we see that

\[ \varphi_2 = \frac{M_0}{2} \left( \sqrt{a^2 + (z + L/2)^2} - |z + L/2| \right). \]

Let us now determine the magnetic field. We know that \( H_z = -\partial_t \varphi_M = -\partial_t (\varphi_1 + \varphi_2) \). Notice that inside the cylinder \( \varphi_1 + \varphi_2 \) contains an additional \( M_0z \) relative to when \( |z| > L^2 \). Computing this trivial derivative, we see that

\[ H_z = -\frac{M_0}{2} \left( \frac{z + L/2}{(a^2 + (z + L/2)^2)^{1/2}} + \frac{z - L/2}{(a^2 + (z - L/2)^2)^{1/2}} \right), \]

\[ = \begin{cases} 0 & |z| > L^2 \\ M_0 & |z| \leq L^2 \end{cases} \]

Now, notice that \( B_z = \mu (H_z + M_0) \) inside the cylinder and \( B_z = \mu H_z \) outside the cylinder. Therefore, the additional ambiguity is cancelled and we see that, both inside and outside the cylinder,

\[ \therefore B_z = -\frac{\mu M_0}{2} \left( \frac{z + L/2}{(a^2 + (z + L/2)^2)^{1/2}} + \frac{z - L/2}{(a^2 + (z - L/2)^2)^{1/2}} \right). \]

**Problem 5.20**

Starting from Jackson’s problem (5.12) and the fact the magnetization \( M \) inside a volume \( \Omega \) bounded by a surface \( \partial \Omega \) is equivalent to a volume current density \( J_M = (\nabla \times M) \) and a surface current density \( (M \times n) \), we are to show that in the absence of macroscopic conduction currents, the total magnetic force on the body can be written

\[ F = -\int_{\Omega} (\nabla \cdot M) B_e \, d^3x + \int_{\partial \Omega} (M \cdot n) B_e \, da, \]

where \( B_e \) is the applied magnetic induction (not including the body in question).

We will make use of a large number of trivial identities listed in the inside cover of Jackson’s text and elsewhere. Our derivation begins with the expression

\[ F = -\int_{\Omega} J_M \times B_e \, d^3x, \]

and using the form of \( J \) given in the problem, we can write

\[ F = \int_{\Omega} (\nabla \times M) \times B_e \, d^3x + \int_{\partial \Omega} (M \times n) \times B_e \, da. \]

In the first integrand, we have the expression \( (\nabla \times M) \times B_e = -B_e \times (\nabla \times M) \). Now, we know that

\[ -B_e \times (\nabla \times M) = (M \cdot \nabla) B_e + (B_e \cdot \nabla) M + M \times (\nabla \times B_e) - \nabla (M \cdot B_e). \]

Notice that the curl of \( B_e \) vanishes. Along similar lines, we see that

\[ (M \times n) \times B_e = -B_e \times (M \times n) = -(B \cdot n) M + (M \cdot B_e) n. \]

\[ ^2 \text{In the other two cases, above and below the cylinder, } \varphi_1 + \varphi_2 \text{ does not depend linearly on } z \text{ and the constant factor will vanish upon differentiation.} \]