Problem 5.18

Let us consider a circular loop of wire having radius $a$ and carrying current $I$ located in vacuum with its center a distance $d$ away from a semi-infinite slab of permeability $\mu$.

a) We are to find the force acting on the loop in the case when the plane of the loop is parallel to the face of the slab.

It is clear that this problem is equivalent to one with the semi-infinite slab replaced by an appropriately chosen image current located a distance $d$ in the $(-z)$-direction. Before going through the extraneous effort to find this image current, we notice that this problem is solved in rather strong generality in problem (5.17); quoting the result, we see that the image current $I' = \frac{\mu - 1}{\mu + 1}I$ and will be located symmetrically in the $(-z)$-direction.

Because the force is nothing other than $F = I' \int d\ell \times \mathbf{B}$, the problem of computing the force reduces to finding the magnetic inductance by the ring of image current. In a spherical coordinate system centered on the original current and obviously oriented, with the center of the image current located at $(2d, 0, 0)$, we see that the image current is located at $r = \sqrt{a^2 + 4d^2}$ and $\theta = \arccos\left(\frac{2d}{\sqrt{a^2 + 4d^2}}\right)$.

In section 5 of Jackson’s text, the magnetic inductance of such a current distribution is completely worked out. Its two nonzero components, $r$ and $\theta$-directions, are given by (in the location of the current loop)

\[
\begin{align*}
B_r &= \frac{\mu_0 I a}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (2n + 1)!!}{2^n n!} \frac{a^{2n+1}}{(a^2 + 4d^2)^{n+3/2}} P_{2n+1} \left(\frac{2d}{\sqrt{a^2 + 4d^2}}\right), \\
B_\theta &= -\frac{\mu_0 I a}{4} \sum_{n=0}^{\infty} \frac{(-1)^n (2n + 1)!!}{2^n (n+1)!} \frac{a^{2n+1}}{(a^2 + 4d^2)^{n+3/2}} P_{1}^{2n+1} \left(\frac{2d}{\sqrt{a^2 + 4d^2}}\right).
\end{align*}
\]

Therefore, we can compute the force acting on the current directly.

\[
\begin{align*}
F &= I' \int d\ell \times \mathbf{B}, \\
&= I' \left( \int d\ell \left( \mathbf{B}_r \hat{\theta} - d\ell \mathbf{B}_\theta \hat{r} \right) \right), \\
&= I' \left( \mathbf{B}_r \int \hat{r} d\ell - \mathbf{B}_\theta \int \hat{\theta} d\ell \right), \\
&= -2\pi a I' \left( \sin \theta \mathbf{B}_r + \cos \theta \mathbf{B}_\theta \right) \hat{z}, \\
&= -\frac{2\pi I' a^2}{\sqrt{a^2 + 4d^2}} \left( a \mathbf{B}_r + 2d \mathbf{B}_\theta \right) \hat{z}
\end{align*}
\]

Putting this all together, we see that

\[
\therefore \mathbf{F} = -\frac{\mu_0 (\mu - 1) I^2 a^2}{\mu + 1} \sum_{n=0}^{\infty} \frac{(-1)^n (2n + 1)!!}{2^n} \frac{a^{2n+1}}{(a^2 + 4d^2)^{n+2}} \left[ \frac{a}{n!} P_{2n+1} \left(\frac{2d}{\sqrt{a^2 + 4d^2}}\right) - \frac{d}{(n+1)!} P_{1}^{2n+1} \left(\frac{2d}{\sqrt{a^2 + 4d^2}}\right) \right] \hat{z}.
\]

b) We are to find the force acting on the loop in the case where the plane of the loop is perpendicular to the face of the slab.

I did it quite wrong the first time; Looks like a mess to fix up; I need my sleep. Sorry Ben.
c) Let us assess the above in the limit where \( d \gg a \).

Let us begin our analysis with the force derived in part a above. The limit where \( d \gg a \) is equivalent to the limit where \( a \to 0 \) and yet \( d/a \) remains constant. Therefore, it is clear that this limit is obtained by keeping only the lowest order in \( a \). Specifically, we see that

\[
\mathbf{F} \approx -\frac{\mu_0 (\mu - 1)}{\mu + 1} I^2 \pi a^2 \left[ \frac{a}{16d^4} (a \cos \theta + d \sin \theta) \right] \hat{z},
\]

\[
= -\frac{\mu_0 (\mu - 1)}{\mu + 1} I^2 \pi a^2 \frac{a}{16d^4} \left( \frac{2d}{\sqrt{4d^2}} + \frac{a}{\sqrt{4d^2}} \right) \hat{z},
\]

\[
= -\frac{\mu_0 (\mu - 1)}{\mu + 1} I^2 \pi a^2 \frac{a}{16d^4} \left( a + \frac{a}{2} \right) \hat{z},
\]

\[
\therefore \mathbf{F} \approx -\frac{\mu_0 (\mu - 1)}{\mu + 1} 3I^2 \pi a^4 \frac{a}{36d^4} \hat{z}.
\]

Notice that this limit is the same as the requirement that the circular current is an idealized magnetic dipole with dipole moment \( m = I\pi a^2 \hat{z} \) (and that the image current is \( m' = I'\pi a^2 \hat{z} \). From section seven of Jackson’s text, we know that the potential energy of dipole \( m \) due to \( m' \) is given by \( U = -m \cdot B \), where \( B \) is the magnetic inductance of a dipole \( m' \) located at \((2d, 0, 0)\). From section (5.6), this is known to be

\[
B = \frac{\mu_0}{4\pi} \frac{3\hat{z} \cdot m' - m'}{(2d)^3} = \frac{\mu_0}{2\pi} \frac{m'}{8d^3} \hat{z}.
\]

Therefore, we have

\[
U = -m \cdot B = -\frac{\mu_0}{2\pi} \frac{m m'}{8d^3},
\]

\[
= \frac{\mu_0}{16\pi d^3} I' \pi a^4,
\]

\[
= \frac{\mu_0 (\mu - 1)}{\mu + 1} \frac{1}{16d^3} I^2 \pi a^4.
\]

Now, to find the force, we differentiate w.r.t. \( 2d \) (because of our earlier assumptions) and find that

\[
\mathbf{F} = -\nabla_{(2d)} U = -3\frac{\mu_0 (\mu - 1)}{\mu + 1} \frac{1}{32d^3} I^2 \pi a^4.
\]

Therefore, we see that

\[
\therefore \mathbf{F} \approx -\frac{\mu_0 (\mu - 1)}{\mu + 1} 3I^2 \pi a^4 \frac{1}{36d^4} \hat{z}.
\]

Problem 5.20 (copied from last week’s homework)

Starting from Jackson’s problem (5.12) and the fact the magnetization \( \mathbf{M} \) inside a volume \( \Omega \) bounded by a surface \( \partial \Omega \) is equivalent to a volume current density \( \mathbf{J}_M = (\nabla \times \mathbf{M}) \) and a surface current density \( (\mathbf{M} \times \mathbf{n}) \), we are to show that in the absence of macroscopic conduction currents, the total magnetic force on the body can be written

\[
\mathbf{F} = -\int_{\Omega} (\nabla \cdot \mathbf{M}) \mathbf{B}_e \, d^3x + \int_{\partial \Omega} (\mathbf{M} \cdot \mathbf{n}) \mathbf{B}_e \, da,
\]

where \( \mathbf{B}_e \) is the applied magnetic induction (not including the body in question).

We will make use of a large number of trivial identities listed in the inside cover of Jackson’s text and elsewhere. Our derivation begins with the expression

\[
\mathbf{F} = -\int_{\Omega} \mathbf{J}_M \times \mathbf{B}_e \, d^3x,
\]