It is clear that when we take the cross product of this with $\mathbf{J} \times \mathbf{B}$, we will obtain a horrendous expression of trigonometric functions. Luckily, as in the case above, many of these will automatically vanish under integration over $\varphi$. Despite our noble intentions, we decided to evaluate this using a computer algebra package. After a surprisingly long amount of computation, it determined that

$$N = \int_0^{2\pi} x(\varphi) \times (J(\varphi) \times B(\varphi))d\varphi = aIB_0\pi \begin{pmatrix} -\cos \theta_0 \\ \cos \theta_0 \\ \sin \theta_0 (\cos \phi_0 - \sin \phi_0) \end{pmatrix}.$$

Now, we remark that based on this calculation, it is only important that the magnetic dipole moment $\mathbf{m}$ remain unchanged. This will be the case for any planar loop of current lying in the same plane as our circle with equal area. Therefore, the shape is irrelevant as long as the area remains constant and the figure lies in the same plane.

**Problem 5.13**

Let us consider a sphere of radius $a$ which carries a uniform surface charge distribution $\sigma(\theta)$. If the sphere is rotated at constant angular velocity $\omega$, then the associated current will induce a magnetic potential. We are to determine the vector potential inside and outside the sphere.

Confined to the surface of the sphere (although we are leaving out the $\delta$-functions for brevity), we see that the current will be

$$\mathbf{J}(\theta) = \rho \mathbf{v} = \sigma \omega \sin \theta \hat{\phi}.$$

Now, following Jackson’s discussion near equation (5.32), we know that the vector potential can be expressed

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3x' = \frac{\mu_0}{4\pi} \sigma \omega a^3 \int \frac{\sin \theta' \hat{\phi}'}{|\mathbf{r} - \mathbf{r}'|} d\Omega.$$

We can expand $1/|\mathbf{r} - \mathbf{r}'|$ of the integrand in terms of the spherical harmonics. Doing this, we see that

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \sigma \omega a^3 \int \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{\ell+1} \frac{\ell}{r_+^{\ell+1}} Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi') [\sin \theta' - \sin \phi' \hat{x} + \cos \phi' \hat{y}] d\Omega.$$

After a lot of initial frustration, it becomes clear that the the expression in square-brackets can be expressed in terms of the spherical harmonics too. Therefore, by the orthogonality conditions, only one term will remain in the sum. Notice that

$$Y_{11}(\theta, \phi) = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{-\phi},$$

and therefore,

$$\sin \theta' \sin \phi' = -\sqrt{\frac{8\pi}{3}} \text{Im} [Y_{11}(\theta', \phi')] \quad \text{and} \quad \sin \theta' \cos \phi' = -\sqrt{\frac{8\pi}{3}} \Re [Y_{11}(\theta', \phi')] .$$

Hence, we can compute directly,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0 \sigma \omega a^3}{4\pi} \int \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{\ell+1} \frac{\ell}{r_+^{\ell+1}} Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi') [\sin \theta' - \sin \phi' \hat{x} + \cos \phi' \hat{y}] d\Omega,$$

$$= -\frac{\mu_0 \sigma \omega a^3}{3} \left\{ \int \frac{r_+^l}{(r_+)^{\ell+1}} Y_{\ell m}(\theta, \phi) Y_{\ell m}^*(\theta', \phi') \left\{ -\text{Im} [Y_{11}(\theta', \phi')] \hat{x} + \Re [Y_{11}(\theta', \phi')] \hat{y} \right\} d\Omega, \right. \right.$$

$$= -\frac{1}{3} \frac{\mu_0 \sigma \omega a^3}{3} \int \frac{r_+^l}{r_+^{\ell+1}} Y_{11}(\theta, \phi) Y_{11}^*(\theta', \phi') \left\{ -\text{Im} [Y_{11}(\theta', \phi')] \hat{x} + \Re [Y_{11}(\theta', \phi')] \hat{y} \right\} d\Omega, \right.$$

$$= \frac{1}{3} \frac{\mu_0 \sigma \omega a^3}{r_+^{\ell+1}} \sin \theta (\sin \phi \hat{x} + \cos \phi \hat{y}),$$

$$\mathbf{A}(\mathbf{r}) = \frac{1}{3} \mu_0 \sigma \omega a^3 \frac{r_+^l}{r_+^{\ell+1}} \sin \theta \hat{\phi}.$$
Inside

Inside the sphere, we have that $r_+ = a$ and $r_- = |r|$. Inserting this directly into the expression above, we see that

$$A(r) = \frac{1}{3} \mu_0 \sigma \omega a \sin \theta \hat{\phi}.$$ 

The magnetic flux density is simply the curl of the vector potential, which is trivially computed to be

$$B(r) = \nabla \times A(r),$$

$$= \frac{1}{3} \mu_0 \sigma \omega \left[ \left( \frac{1}{r} \sin \theta \left( \frac{\partial}{\partial \theta} \left( r \sin^2 \theta \right) \right) \right) \hat{r} - \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \sin \theta \right) \right) \hat{\theta} \right],$$

$$= \frac{1}{3} \mu_0 \sigma \omega \left[ \left( \frac{1}{r} \sin \theta \right) 2r \sin \theta \cos \theta \right] \hat{r} - 2 \sin \theta \hat{\theta},$$

$$\therefore B_{\text{int}}(r) = \frac{2}{3} \mu_0 \sigma \omega \left( \cos \theta \hat{r} - \sin \theta \hat{\theta} \right).$$

Outside

Outside the sphere, we have that $r_+ = |r|$ and $r_- = a$. Inserting this directly into the expression above, we see that

$$A(r) = \frac{1}{3} \mu_0 \sigma \omega a \frac{1}{r^2} \sin \theta \hat{\phi}.$$ 

The magnetic flux density is simply the curl of the vector potential, which is trivially computed to be

$$B(r) = \nabla \times A(r),$$

$$= \frac{1}{3} \mu_0 \sigma \omega a^4 \frac{1}{r^2} \sin \theta \hat{\phi},$$

$$= \frac{1}{3} \mu_0 \sigma \omega a^4 \left[ \left( \frac{1}{r} \sin \theta \left( \frac{\partial}{\partial \theta} \left( r \sin^2 \theta \right) \right) \right) \hat{r} - \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r^2 \sin \theta \right) \right) \hat{\theta} \right],$$

$$= \frac{1}{3} \mu_0 \sigma \omega a^4 \left[ \left( \frac{1}{r} \sin \theta \right) 2r \sin \theta \cos \theta \right] \hat{r} + \sin \theta \hat{\theta},$$

$$\therefore B_{\text{int}}(r) = \frac{1}{3} \mu_0 \sigma \omega a^4 \left( 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right).$$

Problem 5.15

Let us consider two long, straight, parallel wires separated by a distance $d$, carrying current $I$ in opposite directions. We are to describe the magnetic field $H$ in terms of a magnetic scalar potential $\varphi_M$ where $H = -\nabla \varphi_M$.

a) In the limit of $d \to 0$, we are to show that the two-dimensional dipole approaches

$$\varphi_M \approx -\frac{Id \sin \phi}{2\pi \rho} + O\left( \frac{d^2}{\rho^2} \right),$$

where $\rho$ and $\phi$ are the standard polar coordinates.

From our work in class, we know that the magnetic scalar potential of a single infinitely straight wire along the $z$-axis is simply given by $\varphi_M = -\frac{I_0}{2\pi}$. This is also obvious from Ampère’s law. Therefore, because the scalar potentials will linearly superimpose, the potential from the two wires is simply given by

$$\varphi_M = \frac{I}{2\pi} \left( \phi_2 - \phi_1 \right),$$

where $\phi_1$ ($\phi_2$) is the polar angle from wire 1 (2). However, this is not quite adequate for our discussion because we are working in a frame where neither of the two wires are collinear with the $z$-axis. Therefore, we should express this result in terms of the total polar angle $\phi$. 

\(\text{tieaccentlowercase}\)