2.8 Let us consider two parallel, straight line charges separated by a distance $R$ and with equal and opposite linear charge densities $\pm \lambda$.

a) Let us find the surfaces of constant potential. We will show that these are circular cylinders.

With the anticipation only available in hindsight, let us work with polar coordinates—suppressing the longitudinal direction—defined such that the origin is located at the center of one of the circles of constant potential. Specifically, let us say that the first line charge is located at $(\eta, 0)$ and the second at $(R + \eta, 0)$. Although we do not yet know the displacement $\eta$ from the first line charge, we know that if $\eta$ is properly specified, the potential should be independent of the angular coordinate.

In general, the potential will be the linear superposition of the potentials of each of the two line-charges. Having calculated this in homework one, we see

$$\varphi(\rho, \theta) = \frac{\lambda}{4\pi\epsilon_0} \left[ \log(\eta^2 - 2\eta\rho \cos \theta + \rho^2) - \log((R + \eta)^2 - 2(R + \eta)\rho \cos \theta + \rho^2) \right].$$

The requirement that $\varphi$ be constant on a cylinder centered at the origin is equivalent to the condition that $\frac{\partial \varphi}{\partial \theta} = 0$. Using this identity, we compute

$$\frac{\partial \varphi}{\partial \theta} = 0 = \frac{\lambda}{2\pi\epsilon_0} \left( \frac{\eta \rho \sin \theta}{\eta^2 - 2\eta\rho \cos \theta + \rho^2} - \frac{(R + \eta) \rho \sin \theta}{(R + \eta)^2 - 2(R + \eta)\rho \cos \theta + \rho^2} \right),$$

$$= \frac{\lambda \rho \sin \theta}{2\pi\epsilon_0} \left( \frac{\eta}{\eta^2 - 2\eta\rho \cos \theta + \rho^2} - \frac{(R + \eta)}{(R + \eta)^2 - 2(R + \eta)\rho \cos \theta + \rho^2} \right),$$

$$\therefore \eta^2 - 2\eta\rho \cos \theta + \rho^2 = \frac{(R + \eta)^2 - 2(R + \eta)\rho \cos \theta + \rho^2}{(R + \eta)^2 - 2(R + \eta)\rho \cos \theta + \rho^2}.$$

Expanding and collecting terms, we arrive at the constraint

$$\rho^2 = \eta(\eta + R).$$

We must now determine the radius $\rho$ such that $\varphi(\rho, \theta) = V \forall \theta$. To do this, we will insert the expression derived above for $\rho^2$ into our expression for the potential with the condition that $\varphi = V$. Because this will be independent of the angle, let us choose $\theta = \pi/2$ to simplify our expressions. We will need to remember this choice later.

Computing directly, we see that

$$\varphi(\rho, \theta) = V = \frac{\lambda}{4\pi\epsilon_0} \log \left( \frac{\eta^2 - 2\eta\rho \cos \theta + \rho^2}{(R + \eta)^2 - 2(R + \eta)\rho \cos \theta + \rho^2} \right),$$

$$= \frac{\lambda}{4\pi\epsilon_0} \log \left( \frac{\eta^2 + \eta^2 + \eta R}{(R + \eta)^2 + \eta^2 + \eta R} \right),$$

$$= \frac{\lambda}{4\pi\epsilon_0} \log \left( \frac{2\eta^2 + \eta R}{2\eta^2 + 3\eta R + R^2} \right).$$

By rearranging terms, exponentiating, and simplifying, we obtain the quadratic expression

$$\eta^2 \left( 1 - e^{4\pi\epsilon_0 V/\lambda} \right) + \eta R \left( 1 - 3e^{4\pi\epsilon_0 V/\lambda} \right) - R^2 e^{4\pi\epsilon_0 V/\lambda} = 0.$$

Simply using the quadratic formula and a bit of algebra, we see that

$$\eta = -\frac{R}{2(1 - e^{4\pi\epsilon_0 V/\lambda})} \left( 1 - 3e^{4\pi\epsilon_0 V/\lambda} \pm 1 \pm e^{4\pi\epsilon_0 V/\lambda} \right),$$

$$\therefore \eta = \frac{R e^{4\pi\epsilon_0 V/\lambda}}{1 - e^{4\pi\epsilon_0 V/\lambda}} = \frac{R}{e^{-4\pi\epsilon_0 V/\lambda} - 1} = -\frac{Re^{2\pi\epsilon_0 V/\lambda}}{2\sinh(2\pi\epsilon_0 V/\lambda)} \text{ or } \eta = -R/2.$$

Notice that the solution of the quadratic, $\eta = -R/2$, simply demonstrates that between the two wires the potential is constant along the line $(\rho, \pi/2)$. We could have anticipated this solution because we simplified our work to determine $\eta$ using the condition that at $\rho$, $\varphi(\rho, \pi/2) = \varphi(\rho, -\pi/2) = V$. The fact that this cannot be a solution is evident from the last line of our expression for $\varphi(\rho, \theta)$ above: the numerator in the logarithm vanishes if $\eta = -R/2$ and hence this is not a physical solution.
Using our expression for $\rho$, we have

$$\rho^2 = R^2 \frac{e^{8\pi\epsilon_0 V/\lambda}}{(1 - e^{4\pi\epsilon_0 V/\lambda})^2} + R^2 \frac{e^{4\pi\epsilon_0 V/\lambda}}{(1 - e^{4\pi\epsilon_0 V/\lambda})^2} = R^2 \frac{e^{4\pi\epsilon_0 V/\lambda}}{(1 - e^{4\pi\epsilon_0 V/\lambda})^2},$$

therefore, we have shown that if

\[ \rho = R \frac{1}{1 - e^{4\pi\epsilon_0 V/\lambda}} = \frac{Re^{-2\pi\epsilon_0 V/\lambda}}{e^{-4\pi\epsilon_0 V/\lambda} - 1} = -\frac{R}{2\sinh(2\pi\epsilon_0 V/\lambda)}. \]

the potential along a right circular cylinder in the longitudinal direction of radius $\rho$ positioned precisely $\eta$ to the left of the first line-charge is constant and equal to $V$.

It is important to notice that the sign of $\eta$ is ‘somewhat’ important. In particular, if $V/\lambda < 0$ then $\eta$ is a positive displacement to the left of and encloses the first line-charge. However, if $V/\lambda > 0$, then $\eta$ is a negative displacement to the left of the first line-charge; in particular, $-\eta > R$ and so the cylinder is displaced to the right of and encloses the second line-charge.

**b)** We are to demonstrate that the capacitance per unit length of two cylindrical conductors with radii $a$ and $b$ separated by a distance $d$ is given by

$$C = \frac{2\pi\epsilon_0}{\arccosh \left( \frac{d^2 - a^2 - b^2}{2ab} \right)}.$$

**proof:** We know from the definition of capacitance that it is given by $C/L = \frac{\lambda}{V_+ - V_-}$. We must show that the above equation is consistent with this fact.

First, we know that the required problem is equivalent to one in which there are only two line charges, separated by a distance $R = \eta_- + \eta_+$ with radii such that the two cylinders are at constant voltages $V_+$ and $V_-$, respectively. Specifically, we see, reorganizing the expressions derived above,

$$d = \eta_- - \eta_+ = \frac{R}{(e^{-4\pi\epsilon_0 V_-/\lambda} - 1)} - \frac{R}{(e^{-4\pi\epsilon_0 V_+/\lambda} - 1)},$$

and,

$$a = \frac{Re^{-2\pi\epsilon_0 V_+/\lambda}}{(e^{-4\pi\epsilon_0 V_-/\lambda} - 1)} \quad \text{and} \quad b = \frac{Re^{-2\pi\epsilon_0 V_-/\lambda}}{(e^{-4\pi\epsilon_0 V_-/\lambda} - 1)}.$$

Looking at the equation we must verify, it may be helpful to compute $\frac{d^2 - a^2 - b^2}{2ab}$ and see if it points us toward the solution. Let us first compute the numerator. We find that

$$d^2 - a^2 - b^2 = \left( \frac{R^2}{e^{-4\pi\epsilon_0 V_-/\lambda} - 1} \right)^2 \frac{2R^2}{(e^{-4\pi\epsilon_0 V_-/\lambda} - 1)(e^{-4\pi\epsilon_0 V_-/\lambda} - 1)} + \frac{R^2}{(e^{-4\pi\epsilon_0 V_-/\lambda} - 1)^2} - \frac{R^2 e^{-4\pi\epsilon_0 V_-/\lambda}}{(e^{-4\pi\epsilon_0 V_-/\lambda} - 1)^2} \cdot$$

$$= R^2 \left( \frac{1 - e^{-4\pi\epsilon_0 V_-/\lambda}}{e^{-4\pi\epsilon_0 V_-/\lambda} - 1} \right)^2 + \frac{1 - e^{-4\pi\epsilon_0 V_-/\lambda}}{e^{-4\pi\epsilon_0 V_-/\lambda} - 1} \frac{2}{(e^{-4\pi\epsilon_0 V_-/\lambda} - 1)(e^{-4\pi\epsilon_0 V_-/\lambda} - 1)} \right),$$

$$= \frac{R^2}{(e^{-4\pi\epsilon_0 V_-/\lambda} - 1)^2} \left( e^{-4\pi\epsilon_0 V_-/\lambda} - e^{-4\pi\epsilon_0 V_-/\lambda} \right).$$

Now, because

$$2ab = \frac{2R^2 e^{-2\pi\epsilon_0 (V_+ - V_-)/\lambda}}{(e^{-4\pi\epsilon_0 V_-/\lambda} - 1)(e^{-4\pi\epsilon_0 V_-/\lambda} - 1)},$$

we have that

$$\frac{d^2 - a^2 - b^2}{2ab} = \frac{1}{2} \left( e^{-2\pi\epsilon_0 (V_+ - V_-)/\lambda} \left( e^{-4\pi\epsilon_0 V_-/\lambda} + e^{-4\pi\epsilon_0 V_-/\lambda} \right),

= \frac{1}{2} \left( e^{2\pi\epsilon_0 (V_+ - V_-)/\lambda} + e^{-2\pi\epsilon_0 (V_+ - V_-)/\lambda} \right),$$
\[ \therefore \frac{d^2 - a^2 - b^2}{2ab} = \cosh \left( \frac{2\pi \epsilon_0}{\lambda} (V_+ - V_-) \right). \]

Therefore, we see that the capacitance per unit length agrees with the desired formula:

\[
C = \frac{2\pi \epsilon_0}{\text{arccosh} \left( \frac{d^2 - a^2 - b^2}{2ab} \right)} = \frac{2\pi \epsilon_0}{2\pi \epsilon_0 (V_+ - V_-)} = \frac{\lambda}{V_+ - V_-}.
\]

**c)** Let us verify that the expression above for the capacitance agrees with that derived in homework one for a similar problem in the limit where \(d^2 >> a^2 + b^2\).

**proof:** Similar to our previous results, we can make extraordinary progress by simply guessing the form of the answer. First, it should be true that

\[ \text{arccosh} \left( \frac{d^2 - a^2 - b^2}{2ab} \right) = \log \xi, \]

for some \(\xi\). In fact, because we are simply verifying a result, we could presume it to be true and then it would be obvious what \(\xi\) must be. However, let us simply see if such an equation makes sense.

By taking the hyperbolic cosine of each side, we effectively exponentiate the logarithm. Therefore, \(\xi\) must satisfy

\[ \frac{d^2 - a^2 - b^2}{2ab} = \frac{1}{2} (e^{\log \xi} + e^{-\log \xi}) = \frac{1}{2} \left( \xi + \frac{1}{\xi} \right) = \frac{\xi^2 + 1}{2\xi}. \]

This reduces to the following quadratic equation,

\[ \xi^2 - \xi \left( \frac{d^2 - a^2 - b^2}{ab} \right) + 1 = 0. \]

It is clear that in the limit where \(\frac{d^2 - a^2 - b^2}{ab} >> 1\), the solutions approach the trivial solution and that given by

\[ \xi \approx \frac{d^2 - a^2 - b^2}{ab}. \]

Furthermore, in the limit of \(d^2 >> a^2 + b^2\), this approaches \(\xi \sim \frac{d^2}{ab}\).

Therefore, we see that

\[ C = \frac{\pi \epsilon_0}{\text{arccosh} \left( \frac{d^2 - a^2 - b^2}{2ab} \right)} \approx \frac{\pi \epsilon_0}{\text{log} \left( \frac{d^2 - a^2 - b^2}{ab} \right)} \sim \frac{\pi \epsilon_0}{\text{log} \left( \frac{d}{\sqrt{ab}} \right)} \sim \frac{\pi \epsilon_0}{2 \text{log} \left( \frac{d}{\sqrt{ab}} \right)}. \]

\[ \therefore C \sim \frac{2\pi \epsilon_0}{\text{log} \left( \frac{d}{\sqrt{ab}} \right)}. \]