b) Let us compute the charge distribution induced on the sphere.

We call on the simple symmetry of the charge-image-charge system noting that this is identical to the charge distribution induced by the image charge on the outside of the sphere. Because we have already derived this expression in class—and in Jackson—up to a redefinition of \( r \) and \( r' \), we simply have that

\[
\sigma(x) = -\epsilon_0 \frac{\partial \varphi}{\partial x} \bigg|_{|x|=a} = \frac{q}{4\pi a r} \frac{1 - \frac{a^2}{r^2}}{1 + \frac{a^2}{r^2} - 2\frac{a}{r} \cos \theta}^{3/2},
\]

where \( \theta \) is the angle between \( r \) and \( x \).

c) Let us compute the force acting on the charge \( q \).

Using our results above, we see that

\[
F = \frac{q^2}{4\pi \epsilon_0 (a^2 - r^2)^2} \hat{r}.
\]

d) We are to discuss how the work above is altered, if at all, if the sphere were kept at fixed potential \( V \) or if there were total charge \( Q \) on its inner and outer surfaces.

Neither of the two situations alters the work above because neither would effect the interior of the sphere—only the outside. By Gauß’ law, we know that the electric field inside a charged or fixed-potential sphere is identically zero. Because electrostatics is linear, the field inside the sphere will be the linear sum of that described above and that caused by the sphere—which is vanishing. Hence, there is no alteration.

2.7 Let us consider the space \( \mathbb{R}^3 \) satisfying Dirichlet boundary conditions on the plane \( \partial \mathbb{R}^3 \).

a) We are to find the appropriate Green’s function describing this system.

In many ways, this problem is similar to that describing a point charge and an infinite conducting plane. Specifically, we see that the Green’s function given by

\[
G(x, x') = \frac{1}{((x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2)^{1/2}} - \frac{1}{((x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 + x'_3)^2)^{1/2}},
\]

is of the correct form and satisfies the Dirichlet boundary conditions. In particular, we manifestly have that \( G(x, x') = 0 \) \( \forall x' \neq x \). Hence, this is our required Green’s function.

b) Let us say that the potential on the plane \( x'_3 = 0 \) is specified to be \( \varphi = V \) inside a circle of radius \( a \) and vanish outside the circle. We are to find an integral expression for the potential in cylindrical coordinates.

In general, we know that the potential function for a problem with a Green’s function satisfying Dirichlet boundary conditions is given by

\[
\varphi(x) = \frac{1}{4\pi \epsilon_0} \int_{\mathbb{R}^3} \rho(x') G(x, x') d^3 x' - \frac{1}{4\pi} \int_{\partial \mathbb{R}^3} \varphi(x') \frac{\partial G(x, x')}{\partial n} d\alpha',
\]

Because the space is empty of charges, the first integral identically vanishes and we must only consider the boundary integral.

Up to a sign which we will set \( a \) \( \text{a posteriori} \), we see that

\[
\frac{\partial G(x, x')}{\partial x_3} \bigg|_{x'_3=0} = \frac{x_3 - x'_3}{((x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2)^{3/2}} + \frac{x_3 + x'_3}{((x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 + x'_3)^2)^{3/2}} \bigg|_{x'_3=0},
\]

\[
= -\frac{2x_3}{((x_1 - x'_1)^2 + (x_2 - x'_2)^2 + x_3^2)^{3/2}}.
\]

Later, we will see that the \( n' \)-direction should coincide with \( -x_3' \) so that the potential at the surface is positive; this is identical to inserting the seemingly spurious minus sign in the above calculation.
We can now compute the potential. Using our work above, it is clear that
\[
\varphi(\rho, \phi, z) = -\frac{1}{4\pi} \int_{\partial B^3} \varphi(x') \frac{\partial G(x, x')}{\partial n^i} \, da',
\]
\[
= \frac{1}{4\pi} \int_{0}^{2\pi} \int_{0}^{\rho} 2z\rho' d\rho' d\phi'
\]
\[
\varphi(\rho, \phi, z) = \frac{zV}{2\pi} \int_{0}^{2\pi} \int_{0}^{\rho} (\rho' \cos \phi - \rho' \cos \phi')^2 + (\rho' \sin \phi - \rho' \sin \phi')^2 + z^2)^{3/2};
\]
\[
\therefore \varphi(\rho, \phi, z) = \frac{zV}{2\pi} \int_{0}^{2\pi} \int_{0}^{\rho} \rho' d\rho' d\phi'.
\]
\[
\varphi(0, 0, z) = \frac{zV}{2\pi} \int_{0}^{2\pi} \int_{0}^{\rho} \rho' d\rho' d\phi'.
\]
\[
= \frac{zV}{2} \int_{\pi z}^{\rho z} \rho' \cos \phi' d\phi',
\]
\[
= -zV \frac{1}{\sqrt{u}} \left| \int_{\pi z}^{\rho z} \rho' \cos \phi' \right|,
\]
\[
= -zV \left( \frac{1}{\sqrt{u}} \right),
\]
\[
\therefore \varphi(0, 0, z) = V \left( 1 - \frac{z}{\sqrt{u^2 + z^2}} \right).
\]
\[
\text{d) We are to explicitly compute the potential by expanding its expression in the limit where } \rho^2 + z^2 \gg a^2.
\]
Let us define the variable \( \eta^2 \equiv \rho^2 + z^2 \). In general, we can rewrite the potential derived above as
\[
\varphi(\rho, \phi, z) = \frac{zV}{2\pi \eta^3} \int_{0}^{2\pi} \int_{0}^{\rho} \rho' d\rho' d\phi' \left( 1 - \frac{3}{2} \eta^{-2} \left( \rho^2 - 2\rho^2 \cos(\phi - \phi') \right) + \frac{15}{8} \eta^{-4} \left( \rho^2 - 2\rho^2 \cos(\phi - \phi') \right)^2 + O(\eta^{-6}) \right),
\]
\[
\text{When we integrate over the angle } \phi', \text{ all terms independent of } \phi' \text{ will be multiplied by a factor of } 2\pi, \text{ those directly proportional to } \cos(\phi - \phi') \text{ will integrate to zero, and the term proportional to } \cos^2(\phi - \phi') \text{ will obtain a factor of } \pi. \text{ Therefore,}
\]
\[
\varphi(\rho, \phi, z) = \frac{zV}{2\pi \eta^3} \int_{0}^{2\pi} \int_{0}^{\rho} \rho' d\rho' d\phi' \left( 1 - \frac{3}{2} \eta^{-2} \left( \rho^2 - 2\rho^2 \cos(\phi - \phi') \right) + \frac{15}{8} \eta^{-4} \left( \rho^2 - 2\rho^2 \cos(\phi - \phi') \right)^2 + O(\eta^{-6}) \right),
\]
\[
= \frac{zV}{\eta^3} \left( \frac{\rho^2}{2} + \frac{3\rho^4}{8\eta^2} + \frac{15\rho^4}{16\eta^4} + \frac{5\rho^6}{16\eta^6} + O(\eta^{-6}) \right),
\]
\[
= \frac{zV a^2}{2\eta^3} \left( 1 - \frac{3a^2}{4\eta^2} + \frac{5(3\rho^2 a^2 + a^4)}{8\eta^4} + O((\rho^2 + z^2)^{-6}) \right),
\]
\[
\therefore \varphi(\rho, \phi, z) = \frac{zV a^2}{2(\rho^2 + z^2)^{3/2}} \left( 1 - \frac{3a^2}{4(\rho^2 + z^2)} + \frac{5(3\rho^2 a^2 + a^4)}{8(\rho^2 + z^2)^2} + O((\rho^2 + z^2)^{-6}) \right).
\]