2.13 a) We were to compute the interior potential of a cylinder composed of two separated, conducting halves which are kept at potentials \( V_1 \) and \( V_2 \).

From our experience with this problem, we should expect that the potential \( \varphi \) can be expanded as a cosine series over the azimuthal coordinate. Specifically, we expect

\[
\varphi(\rho, \phi) = \alpha_0 + \sum_{m=1}^{\infty} \alpha_m \rho^m \cos(m\phi),
\]

where the coefficients \( \alpha_m \) are determined using the orthogonality conditions. We see that

\[
\alpha_0 = \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} \varphi(b, \phi) d\phi = \frac{1}{2\pi} \pi (V_1 + V_2) = \frac{V_1 + V_2}{2},
\]

and, similarly,

\[
\alpha_m = \frac{1}{\pi b^m} \left[ \int_{-\pi/2}^{\pi/2} V_1 \cos(m\phi) d\phi + V_2 \int_{\pi/2}^{3\pi/2} V_2 \cos(m\phi) d\phi \right] = \frac{(V_1 - V_2) (-1)^m}{\pi m b^m} |m\in(2\mathbb{Z}+1)|.
\]

Therefore, we can use some of Jackson’s tricks in elementary complex analysis to see

\[
\varphi(\rho, \phi) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \sum_{m \in \text{odd}} \frac{(-1)^m}{m} \frac{\rho^m}{b^m} \cos(m\phi),
\]

\[
= \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \text{Im} \left[ \sum_{m \in \text{odd}} i^m \rho^m e^{im\phi} b^m \right],
\]

\[
= \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \text{Im} \left[ \log \left( \frac{1 + i(\rho/b)e^{i\phi}}{1 - i(\rho/b)e^{i\phi}} \right) \right],
\]

\[
= \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \text{arctan} \left( \frac{2\rho/b \cos \phi}{1 - \rho^2/b^2} \right),
\]

\[
\therefore \varphi(\rho, \phi) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \text{arctan} \left( \frac{2\rho \cos \phi}{b^2 - \rho^2} \right).
\]

b) Let us calculate the surface charge density on each half of the cylinder.

We recall that the surface charge density is given by

\[
\sigma(\phi) = -\epsilon_0 \frac{\partial \varphi(\rho, \phi)}{\partial \rho} \bigg|_{\rho=b} = -\epsilon_0 \frac{V_1 - V_2}{\pi} \frac{\partial}{\partial \rho} \text{arctan} \left( \frac{2\rho \cos \phi}{b^2 - \rho^2} \right).
\]

Using a computer algebra package to evaluate the derivative—time becomes too precious to evaluate by hand—we see that

\[
\sigma(\phi) = -\epsilon_0 \frac{V_1 - V_2}{\pi} \frac{2b(2b^2) \cos \phi}{b(1 + \cos(2\phi))},
\]

\[
= -\frac{2\epsilon_0 (V_1 - V_2)}{\pi} \frac{2b \cos 2\phi}{b(1 + \cos(2\phi))},
\]

\[
= -\frac{2\epsilon_0 (V_1 - V_2)}{\pi} \frac{2b \cos^2 \phi}{b},
\]

\[
\therefore \sigma(\phi) = -\frac{\epsilon_0 (V_1 - V_2)}{\pi b \cos \phi}.
\]