4. The Force Between Conductors

a) We are to find the attractive force between the conductors of the parallel plate capacitor described in problem (2.a) and the parallel cylinders of problem (3) for fixed charges on each conductor.

We know that the energy stored between an arbitrary capacitor with fixed charges is given by $Q^2/2C$ where $Q$ is the charge on each conductor and $C$ is the capacitance of the system.

Let us first consider the parallel plate capacitor. Using our results from problem (2.a), we can easily determine that the energy of the capacitor is given by $W = Q^2/2A\varepsilon_0$. Because the force $F = -\frac{\partial W}{\partial d}$, we have that $F = -\frac{Q^2}{2A\varepsilon_0}$.

Similarly, we can use our results from problem (3) for the capacitance of the parallel cylinder system to arrive at the energy stored per unit length, $W = \frac{Q^2}{2\pi\varepsilon_0} \log \left( \frac{d}{\sqrt{a_1a_2}} \right)$.

Therefore, we see that $F = -\frac{Q^2}{2\pi\varepsilon_0d}$.

b) We are to find the attractive force between the conductors of the parallel plate capacitor described in problem (2.a) and the parallel cylinders of problem (3) for fixed potential difference of the conductors.

We know that the energy stored between an arbitrary capacitor with fixed potentials on each conductor is given by $\frac{1}{2}CV^2$ where $V$ is the voltage difference between the two conductors and $C$ is the capacitance of the system.

Let us first consider the parallel plate capacitor. Using our results from problem (2.a), we can determine that the energy of the capacitor is given by $W = \frac{\varepsilon_0 A d}{2\varepsilon_0} V^2$. Because the force $F = -\frac{\partial W}{\partial d}$, we have that $F = -\frac{\varepsilon_0 A}{2d^2} V^2$.

Similarly, we can use our results from problem (3) for the capacitance of the parallel cylinder system to arrive at the energy stored per unit length, $W = \frac{\pi\varepsilon_0}{2\log \left( \frac{d}{\sqrt{a_1a_2}} \right)} V^2$.

Therefore, we see that $F = -\frac{\pi\varepsilon_0}{2\log \left( \frac{d}{\sqrt{a_1a_2}} \right)} V^2$.

5. Thomson’s Theorem

If an empty region is bounded by a number of equipotential surfaces, then the electrostatic energy inside the region is absolutely minimized.

**proof:** Let us consider the energy within the bounded, compact region $\Omega$ with boundary $\partial\Omega$ which is composed of equipotential surfaces. We will show that the electrostatic energy $W$ of the region $\Omega$ is absolutely minimized.

Recall from the discussion in section (1.11) of Jackson’s text that, in general, the electrostatic energy of a region $\Omega$ is given by

$$W = -\frac{\varepsilon_0}{2} \int_\Omega \nabla^2 \phi \, d^3x,$$

where $\phi(x)$ is the scalar potential. Integrating this expression by parts, we see that

$$W = \frac{-\varepsilon_0}{2} \left( \int_{\partial\Omega} \phi \nabla \phi \, da - \int_\Omega |\nabla \phi|^2 \, d^3x \right).$$

Using the definition of the scalar potential $\phi$, $E = -\nabla \phi$, we see that this implies that

$$W = \frac{\varepsilon_0}{2} \int_{\partial\Omega} \phi E \, da + \frac{\varepsilon_0}{2} \int_\Omega |\nabla \phi|^2 \, d^3x.$$