4. **The Force Between Conductors**

a) We are to find the attractive force between the conductors of the parallel plate capacitor described in problem (2.a) and the parallel cylinders of problem (3) for fixed charges on each conductor. We know that the energy stored between an arbitrary capacitor with fixed charges is given by \( Q^2 / 2C \) where \( Q \) is the charge on each conductor and \( C \) is the capacitance of the system.

Let us first consider the parallel plate capacitor. Using our results from problem (2.a), we can easily determine that the energy of the capacitor is given by

\[
W = \frac{Q^2}{2A\varepsilon_0}. \tag{4.1a}
\]

Because the force \( F = -\frac{\partial W}{\partial d} \), we have that

\[
\therefore F = -\frac{Q^2}{2A\varepsilon_0}. \tag{4.1b}
\]

Similarly, we can use our results from problem (3) for the capacitance of the parallel cylinder system to arrive at the energy stored per unit length,

\[
W = \frac{Q^2}{2\pi \varepsilon_0 \log\left(\frac{d}{\sqrt{a_1 a_2}}\right)}. \tag{4.1c}
\]

Therefore, we see that

\[
\therefore F = -\frac{Q^2}{2\pi \varepsilon_0 d}. \tag{4.1d}
\]

b) We are to find the attractive force between the conductors of the parallel plate capacitor described in problem (2.a) and the parallel cylinders of problem (3) for fixed potential difference of the conductors. We know that the energy stored between an arbitrary capacitor with fixed potentials on each conductor is given by \( \frac{1}{2}CV^2 \) where \( V \) is the voltage difference between the two conductors and \( C \) is the capacitance of the system.

Let us first consider the parallel plate capacitor. Using our results from problem (2.a), we can determine that the energy of the capacitor is given by

\[
W = \frac{\varepsilon_0 A}{2d} V^2. \tag{4.2a}
\]

Because the force \( F = -\frac{\partial W}{\partial d} \), we have that

\[
\therefore F = -\frac{\varepsilon_0 A}{2d} V^2. \tag{4.2b}
\]

Similarly, we can use our results from problem (3) for the capacitance of the parallel cylinder system to arrive at the energy stored per unit length,

\[
W = \frac{\pi \varepsilon_0}{2 \log\left(\frac{d}{\sqrt{a_1 a_2}}\right)} V^2. \tag{4.2c}
\]

Therefore, we see that

\[
\therefore F = -\frac{\pi \varepsilon_0}{2 \log\left(\frac{d}{\sqrt{a_1 a_2}}\right)} d V^2. \tag{4.2d}
\]

5. **Thomson’s Theorem**

If an empty region is bounded by a number of equipotential surfaces, then the electrostatic energy inside the region is absolutely minimized.

**proof:** Let us consider the energy within the bounded, compact region \( \Omega \) with boundary \( \partial\Omega \) which is composed of equipotential surfaces. We will show that the electrostatic energy \( W \) of the region \( \Omega \) is absolutely minimized.

Recall from the discussion in section (1.11) of Jackson’s text that, in general, the electrostatic energy of a region \( \Omega \) is given by

\[
W = \frac{-\varepsilon_0}{2} \int_{\Omega} \nabla^2 \phi \, d^3x,
\]

where \( \phi(x) \) is the scalar potential. Integrating this expression by parts, we see that

\[
W = \frac{-\varepsilon_0}{2} \left( \int_{\partial\Omega} \phi \nabla \phi \, da - \int_{\Omega} |\nabla \phi|^2 \, d^3x \right). \tag{5.1a}
\]

Using the definition of the scalar potential \( \phi, E = -\nabla \phi \), we see that this implies that

\[
W = \frac{\varepsilon_0}{2} \int_{\partial\Omega} \phi E \, da + \frac{\varepsilon_0}{2} \int_{\Omega} |\nabla \phi|^2 \, d^3x. \tag{5.1b}
\]
Because the boundary $\partial \Omega$ is composed of equipotential pieces, $\phi(x)$ must be constant on the boundary. Therefore, the first integral can be reduced using Gauss’ law. Specifically, we have that

$$W = \frac{\epsilon_0}{2} \int_{\partial \Omega} \phi E \, da + \frac{\epsilon_0}{2} \int_{\Omega} |\nabla \phi|^2 \, d^3x,$$

$$= \frac{Q_{\text{int}}}{2} \phi|_{\partial \Omega} + \frac{\epsilon_0}{2} \int_{\Omega} |\nabla \phi|^2 \, d^3x,$$

$$= \frac{\epsilon_0}{2} \int_{\Omega} |\nabla \phi|^2 \, d^3x,$$

where we have used the fact that $\Omega$ is empty and therefore $Q_{\text{int}} = 0$. It should be noted that each term is positive definite and, in particular, the first term is absolutely minimized by an equipotential boundary—indeed, it vanishes entirely. However, it remains for us to show that the remaining source of electrostatic energy is minimized.

We will briefly digress to discover what conditions $\phi$ must satisfy so that $W$ is extremized.

Let us imagine that the potential function $\phi$ extremizes the energy $W$. In particular, this means that a first-order variation $\delta \phi$ in the potential function should not vary the energy $W$ (keeping $\delta \phi = 0$ on the boundary).

Therefore, the condition that $W$ is extremized by $\phi$ is equivalent to the requirement that $\delta W = 0$ for any first-order variation $\delta \phi$, fixed on the boundary $\partial \Omega$. In particular, using integration by parts and the fact that $\delta \phi$ vanishes on the boundary, we have that

$$\delta W = 0 = \frac{\epsilon_0}{2} \int_{\Omega} 2 \nabla \phi \cdot \nabla (\delta \phi) \, d^3x,$$

$$= \frac{\epsilon_0}{2} \int_{\Omega} 2 \nabla \phi \cdot \nabla (\delta \phi) \, d^3x,$$

$$= \epsilon_0 \nabla \phi \cdot \delta \phi|_{\partial \Omega} - \epsilon_0 \int_{\Omega} \nabla^2 \phi \delta \phi \, d^3x,$$

$$= -\epsilon_0 \int_{\Omega} \nabla^2 \phi \delta \phi \, d^3x = 0,$$

$$\therefore \nabla^2 \phi = 0.$$

In the last step we have used the fact that because the variation $\delta \phi$ was arbitrary, $\nabla^2 \phi$ must precisely vanish everywhere in $\Omega$.

Therefore, we have shown that the energy is extremized by a potential which satisfies Laplace’s equation. In particular, this implies that the bulk term of the electrostatic energy is extremized by a vacuum interior.

The Laplace equation is satisfied by the region $\Omega$ because it free of charge. Therefore, the second source of electrostatic energy is extremized—hopefully, minimized. We showed above that because the equipotential distribution of charges on the surface components minimizes the first term. Therefore, if a region is bounded by an equipotential surface, the electrostatic energy of that region is minimized.

$\delta \sigma \rho \delta \epsilon \delta \xi \gamma$