BRIEF ANSWERS TO AN OLD FIRST EXAM

1. (a) $24^3 + 24^2 = 14400$.  
(b) $\binom{24}{3} = 24 \cdot 23 \cdot 22 / 3! = 2024$.

2. $\binom{8}{2} \cdot 6 = 8!/(2! \cdot 6!)$ = 1680.

3. (a) $P(E \cap F) \leq P(F) = 0.4$, and equality occurs if $F$ is a subset of $E$.
(b) $P(E \cap F) = P(E) + P(F) - P(E \cup F) \geq 0.7 + 0.4 - 1.0 = 0.1$, and equality can occur.

4. Let $2 + x_i$ denote the number of balls in the $i$th urn. Then $x_1 + x_2 + x_3 + x_4 = 8$ and $x_i \geq 0$. Thus there are $\binom{11}{4} = 11 \cdot 10 \cdot 9 / 3! = 165$ ways.

5. Since $E^c \cap F^c \cap G^c = (E \cup F \cup G)^c$, it follows that $P(F^c \cap G^c \cap E^c) = 1 - P(E \cup F \cup G)$. By the inclusion–exclusion formula we see that this is $1 - [(0.6 + 0.5 + 0.3) + (0.3 + 0.2 + 0.2) - 0.1 = 0.2]$.

6. (a) $S = \{WWW, WWL, WLW, WLL, LWW, LWL, LLW, LLL\}$.  
(b) Let $A$ be the event that they win the first game, $B$ the event that they win at least one game. Then $P(A) = 1/2$ and $P(B) = 7/8$. Hence $P(A|B) = P(AB)/P(B) = (1/2)/(7/8) = 4/7$.

7. Equivalently, a die is rolled until a 6 comes up. What is the probability that the first 6 is on an odd-numbered roll. The probability that the first 6 occurs on the $(2i + 1)$th roll is $(5/6)^{2i}(1/6)$. Hence $\sum_{i=0}^{\infty} (5/6)^{2i}(1/6) = (1/6)(1/(1 - (5/6)^2)) = 6/11$.

8. Since $E \cap F^c$ and $F$ are disjoint events, and since their union is the set $E \cup F$, it follows by the Third Axiom that $P(E \cap F^c) + P(F) = P(E \cup F)$. But $E$ is a subset of $E \cup F$, so $P(E \cup F) \geq P(E)$. Hence $P(E \cup F) = P(E) - P(F)$.


10. The question is whether $P(E)P(F) = P(EF)$. Clearly $P(E) = 3/5$. Also, $P(F) = 5 \cdot (0.6)^3(0.4)^2 = 10(0.6)^3(0.4)^2$. Similarly, $P(EF) = (0.6)(4/5)(0.6)^2(0.4)^2 = 6(0.6)^3(0.4)^2$. Thus the events are independent.

11. The sample space here consists of ordered pairs of distinct cards, of which there are $52 \cdot 51$ possibilities, each one equally likely. Let $S$ denote the event that the two cards have the same rank. After the first card is selected, the second one will have the same rank in 3 out of 51 cases, so $P(S) = 3/51 = 1/17$. Let $A$ be the event that the first card has rank at least as high as the second. Then by Bayes’ formula, $P(A) = P(A|S)P(S) + P(A|S^c)P(S^c)$. But $P(A|S) = 1$, and by symmetry we see that $P(A|S^c) = 1/2$. Hence $P(A) = 1 \cdot 1/17 + (1/2) \cdot (16/17) = 9/17$. 