Brief Answers to the First Sample Second Exam

1. (a) \(E[X] = \int_{-\infty}^{\infty} xf(x) \, dx = \int_{0}^{1} 3x^3 \, dx = 3/4.\)
(b) \(E[X^2] = \int_{-\infty}^{\infty} x^2 f(x) \, dx = \int_{0}^{1} 3x^4 \, dx = 3/5.\) Hence \(\text{Var}(X) = E[X^2] - E[X]^2 = 3/5 - (3/4)^2 = 3/80 = 0.0375.\)

2. (a) Poisson with parameter \(\lambda = 2.\)
(b) Binomial with \(n = 20, p = 1/8.\)
(c) Uniform on \([0, 360).\)

3. (a) \(F_X(x) = 1 - e^{-\lambda x} = 1 - e^{-x/2}, \) so \(P(X < 3) = 1 - e^{-3/2} = 0.7769.\)
(b) The same, by the memoryless property of the exponential variable.

4. \(0.875 = P(X > 0) = P(X - 4 > -4) = P((X - \mu)/\sigma > -4/\sigma) = 1 - \Phi(-4/\sigma) = \Phi(4/\sigma).\) But \(\Phi\) is continuous and strictly increasing with values from 0 to 1, so the equation \(\Phi(x) = 0.875\) has a unique root \(x\), which by the table of values of \(\Phi\) we see is approximately \(x = 1.15.\) Hence \(\sigma = 4/1.15 = 3.478\) and so \(\text{Var}(X) = \sigma^2 = 12.1.\)

5. The variable is binomial with \(n = 10^4\) and \(p = 0.2.\) Hence \(\mu = np = 2000\) and \(\sigma^2 = np(1 - p) = 1600\) so that \(\sigma = 40.\) Now \(P(2040 < X < 2080) = P(40 < X < 80) = P(1 < (X - 2000)/40 < 2),\) so by using a corresponding normal variable we deduce that the above is approximately \(\Phi(2) - \Phi(1) = 0.9772 - 0.8413 = 0.1359.\)

6. (a) If you win on the first bet then you gain $1. If you win on the second bet then your net gain is \(-1 + 2 = 1.\) If you win on the third bet then your net gain is \(-1 - 2 + 4 = 1.\) This pattern continues through winning on the tenth bet. The only alternative is that you lose ten bets in a row, which occurs with probability \(1/2^{10}.\) Thus \(P(X = 1) = 1 - 1/2^{10}.\) In the bad case you lose \(1 + 2 + \cdots + 2^9 = 2^{10} - 1.\) Thus \(P(X = 1 - 2^{10}) = 1/2^{10}.\)
(b) \(P(X > 0) = 1 - 1/2^{10}.\)
(c) \(E[X] = 1 \cdot (1 - 1/2^{10}) + (1 - 2^{10}) \cdot 1/2^{10} = 0.\)

7. Suppose we have two samples to be tested. By strategy (1) this involves 2 tests. Thus the question is for what \(p\) do we expect that using strategy (2) will lead to less than 2 tests, i.e., \(E[X] < 2.\) If neither sample has the disease, then we perform 1 test. The probability of this is \((1 - p)^2.\) Thus \(P(X = 1) = (1 - p)^2.\) Otherwise we perform 3 tests, so that \(P(X = 3) = 1 - (1 - p)^2.\) Hence \(E[X] = (1 - p)^2 + 3(1 - (1 - p)^2) = 1 + 4p - 2p^2.\) This is < 2 if \(0 \leq p < 1 - \sqrt{2} / 2 = 0.29289.\)