Homework 9
Due: Friday, April 10, 2009

Section 17.1, pg. 1096: 15-18, 29-32.
Section 17.2, pg. 1107: 5, 7, 11, 31, 46.
Section 17.3, pg. 1117: 3, 9, 11, 29, 33.
Section 17.4, pg. 1125: 3, 9, 13, 25.
Section 17.5, pg. 1132: 3, 12, 13.

Solutions
17.1 #15-18
15: IV. Constant vectors everywhere;
16: I. Constant vectors if z is fixed;
17: III. Even x, y are zeros, vectors has certain length;
18: II. Length of vectors increase as x, y, z increase from 0.

17.1 #29-32
29: IV. $\langle y, x \rangle$;
30: III. $\langle 2x, -2y \rangle$;
31: II. $\langle 2x, 2y \rangle$;
32: I. $\sqrt{x^2 + y^2}$.

17.2 # 5
C is the parabola $y = x^2$ from (1,1) to (3,9). This can be parameterized as $\vec{r}(t) = (t, t^2)$ with $1 \leq t \leq 3$.

$$\int_C (xy + \ln(x))dy = \int_1^3 (x(t)y(t) + \ln(x(t)))y'(t)dt$$
$$= \int_1^3 (t \cdot t^2 + \ln(t))2tdt$$
$$= \int_1^3 (t^4 + t\ln(t))dt$$
$$= \left[\frac{t^5}{5}\right]_1^3 + \left[\frac{t^2}{4}(2\ln(t) - 1)\right]_1^3$$
$$= \frac{232}{5} + \frac{9\ln(9)}{4}$$

17.2 # 7
There are two segments: $\vec{r}_1(t) = (2t, 0)$ and $\vec{r}_2(t) = (2, 3t)$.

$$\int_C xy \, dx = \int_{c_1} xy \, dx + \int_{c_2} xy \, dx$$
$$= \int_0^1 x_1(t)y_1(t)x_1'(t)dt + \int_0^1 x_2(t)y_2(t)x_2'(t)dt$$
$$= \int_0^1 (2t)(0)(2)dt + \int_0^1 (2)(3t)(0)dt$$
$$= 0$$

$$\int_C (x - y) \, dy = \int_{c_1} (x - y) \, dy + \int_{c_2} (x - y) \, dy$$
\[
\begin{align*}
\int_0^1 (x_1(t) - y_1(t))y'_1(t)dt + \int_0^1 (x_2(t) - y_2(t))y'_2(t)dt \\
= \int_0^1 (2t - 0)(0)dt + \int_0^1 (2 - 3t)(3)dt \\
= \int_0^1 (6 - 9t)dt \\
= 3/2
\end{align*}
\]

\[
f_C \, xy \, dx + (x - y) \, dy = \int_C \, xy \, dx + \int_C \, (x - y) \, dy = 0 + 3/2 = 3/2
\]

**17.2 # 11**
C is just the line \( \vec{r}(t) = \langle t, 2t, 3t \rangle \) with \( 0 \leq t \leq 1 \). \( |\vec{r}'(t)| = \sqrt{1 + 4 + 9} \).

\[
\int_C xe^{yz} \, ds = \int_0^1 te^{2t^3} \, \sqrt{14} \, dt \\
= \sqrt{14} \int_0^1 te^{6t^2} \, dt \\
= \sqrt{14}/12 \int_0^1 e^{6t} \, du \\
= \sqrt{14}/12(e^6 - 1)
\]

**17.2 # 31**
The half-circle is parameterized as \( \vec{r}(t) = 2\langle \cos(t), \sin(t) \rangle \) with \(-\pi/2 \leq t \leq \pi/2\). So \( |\vec{r}'(t)| = 2 \).

\[
m = \int_C \rho(x, y)ds = \int_{-\pi/2}^{\pi/2} 2k \, dt = 2k \pi \\
M_y = \int_C \rho(x, y)x ds = \int_{-\pi/2}^{\pi/2} 2k2 \cos(t) \, dt = 8k \\
M_x = \int_C \rho(x, y)y ds = \int_{-\pi/2}^{\pi/2} 2k2 \sin(t) \, dt = 0
\]

So, the center of mass is at \( (M_y/m, M_x/m) = (8/\pi, 0) \) and the mass is \( 2k\pi \).

**17.2 # 46**
A circle of radius \( r \) is parameterized as \( \vec{r}(t) = r\langle \cos(t), \sin(t) \rangle \), and \( |\vec{r}'(t)| = r \). Thus

\[
\int_C \vec{B} \cdot d\vec{r} = \int_C \vec{B} \, \cos \theta \, ds
\]

where \( \theta \) is the angle between \( \vec{B} \) and the differential tangent vector \( d\vec{r} \). The text reveals that the magnetic field is oriented along the circle, though, so \( \theta = 0 \)

\[
\int_C \vec{B} \cdot d\vec{r} = \int_C B \, ds = \int_0^{2\pi} Br \, dt
\]

The text also hints that \( B \) depends only on the radius from the wire.

\[
\int_0^{2\pi} Br \, dt = Br \int_0^{2\pi} dt = 2\pi Br = \mu_0 I
\]

Solving for \( B \) gets \( B = \frac{\mu_0 I}{2\pi r} \).

**17.3 # 3**
Clear the derivatives are continuous, so check Clairaut’s theorem. \( \frac{\partial P}{\partial y} = 5 \) and \( \frac{\partial Q}{\partial x} = 5 = \frac{\partial P}{\partial y} \). So it is conservative. The potential function is \( f(x, y) = 3x^2 + 5xy + 2y^2 + k \), where \( k \) is some constant.

**17.3 # 9**
\( \frac{\partial P}{\partial y} = e^x + \cos y \) and \( \frac{\partial Q}{\partial x} = e^x + \cos y \). So it is conservative. The potential function is \( f(x, y) = ye^x + x \sin y + k \), where \( k \) is some constant.

**17.3 # 11**

a) \( \vec{F}(x, y) \) is conservative: \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2x \). Thus, it is path independent.
b) We don’t have to use any of the given paths. Take the line \( \vec{r}(t) = \langle 1 + 2t, 2 \rangle \) with \( 0 \leq t \leq 1 \).

\[
\int_C \vec{F} \cdot d\vec{r} = \int_C P \, dx + \int_C Q \, dy \\
= \int_0^1 2x(t)y(t)x'(t) \, dt + 0 \\
= \int_0^1 2(1 + 2t) \cdot 2t \, dt \\
= 8 \int_0^1 (1 + 2t) \, dt \\
= 8(1 + 1) \\
= 16
\]

**17.3 # 29**

It’s all in one piece, so it is connected. There are no holes, so it is simply-connected. The bounds (either 0 or infinity) are not part of the domain itself, so it is open.

**17.3 # 33**

a) \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \)
b) Both curves are given by \( \vec{r}(t) = \langle \cos t, \sin t \rangle \). The top half has \( 0 \leq t \leq \pi \), and the bottom by \( \pi \leq t \leq 2\pi \). Note: for \( C_2 \), we have to progress from \((1,0)\) to \((-1,0)\), so the limits I’ve given are actually for \(-C_2\).

Top path:

\[
\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^\pi P \, dx + \int_0^\pi Q \, dy \\
= \int_0^\pi \frac{-\sin t}{\cos t^2 + \sin t^2} (-\sin t) \, dt + \int_0^\pi \frac{\cos t}{\cos t^2 + \sin t^2} (\cos t) \, dt \\
= \int_0^\pi 1 \, dt \\
= \pi
\]

Bottom path:

\[
\int_{-C_2} \vec{F} \cdot d\vec{r} = \int_\pi^{2\pi} P \, dx + \int_\pi^{2\pi} Q \, dy \\
= \int_\pi^{2\pi} \frac{-\sin t}{\cos t^2 + \sin t^2} (-\sin t) \, dt + \int_\pi^{2\pi} \frac{\cos t}{\cos t^2 + \sin t^2} (\cos t) \, dt \\
= \int_\pi^{2\pi} 1 \, dt \\
= \pi
\]

So \( -\int_{C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} = \pi \). Not path independent.

This is not a contradiction of Theorem 6, since \( P \) and \( Q \) do not have continuous first order derivatives. They are discontinuous at \((0, 0)\).

**17.4 # 3**

a) The line integral has three line segments, \( \vec{r}_1(t) = \langle t, 0 \rangle \), \( \vec{r}_2(t) = \langle 1, 2t \rangle \), \( \vec{r}_3(t) = \langle 1 - t, 2(1 - t) \rangle \). Each will have \( 0 \leq t \leq 1 \).

\[
\int_C xy \, dx = \int_C x(t)y(t)x'(t) \, dt
\]
\[
\int_C x^2y^3 \, dy = \int_C x(t)^2 y(t)^3 y'(t) \, dt
\]
\[
= \int_{C_1} x(t)^2 y(t)^3 y'(t) \, dt + \int_{C_2} x(t)^2 y(t)^3 y'(t) \, dt + \int_{C_3} x(t)^2 y(t)^3 y'(t) \, dt
\]
\[
= \int_0^1 1(0)^3(0) \, dt + \int_0^1 (2t)^3(2) \, dt + \int_0^1 (1 - t)^2 3(1 - t)^3(-2) \, dt
\]
\[
= 0 + 2^4 \int_0^1 t^3 \, dt - 2^4 \int_0^1 (1 - t)^5 \, dt
\]
\[
= 4 - 8/3
\]
So \( \int_C xy \, dx + x^2y^3 \, dy = -2/3 + 4 - 8/3 = 2/3. \)

b) \[
\int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA = \int \int_D (2xy^3 - x) \, dA
\]
\[
= \int_0^1 \int_0^{2x} (2xy^3 - x) \, dy \, dx
\]
\[
= \int_0^1 \left( \frac{2x^5}{3} - 2x^2 \right) \, dx
\]
\[
= 2/3
\]

Of course, they’re the same.

17.4 # 9

\[
\int_C (y + e^{\sqrt{x}}) \, dx + (2x + \cos y^2) \, dy = \int \int_D \frac{\partial}{\partial x} (2x + \cos y^2) - \frac{\partial}{\partial y} (y + e^{\sqrt{x}})
\]
\[
= \int_0^1 \int_{\sqrt{x}}^{2-x} (2-1) \, dy \, dx
\]
\[
= \int_0^1 (\sqrt{x} - x^2) \, dx
\]
\[
= 1/3
\]

17.4 # 13

Note that the path as given is negatively oriented. If we use Green’s Theorem, we have to remember to switch the sign. \( \int_C \vec{F} \cdot d\vec{r} = -\int_{-C} \vec{F} \cdot d\vec{r} = \int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA. \)

\[
\int \int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA
\]
\[
= \int_0^\pi \int_0^{\sin x} (2x - 3y^2) \, dy \, dx
\]
\[
= \int_0^\pi \left[ 2x \sin x \sin x - \sin x^3 \right] \, dx
\]
\[
= 2 \left[ \sin x - x \cos x \right]_0^\pi - \left[ -1/3 \sin x^2 \cos x \right]_0^\pi - 2/3 \int_0^\pi \sin x \, dx
\]
\[
= 2(0 - 0 + \pi - 0) + 0 + 2/3(-1 - 1)
\]
\[
= 2\pi - 4/3
\]
So \( \int_C \vec{F} \cdot d\vec{r} = 4/3 - 2\pi. \)
17.4 # 25
The definition of the moments are
\[ I_y = \int_D x^2 \rho(x,y) dA \] and \[ I_x = \int_D y^2 \rho(x,y) dA. \] Since \( \rho \) is constant, these become \[ I_y = \rho \int_D x^2 dA \] and \[ I_x = \rho \int_D y^2 dA. \] If we can find \( \vec{F}(x,y) = \langle P, Q \rangle \) such that \( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = x^2 \) for \( I_y \), then we can use Green’s Theorem. Choose \( \vec{F} = (0, x^3/3) \). So
\[ I_y = \rho \int_D x^2 dA = \rho \int_C \langle 0, x^3/3 \rangle \cdot d\vec{r} = \rho/3 \int_C x^3 \cdot dy \]
Similarly, for \( I_x \), we choose \( \vec{F} = (-y^3,0) \).
\[ I_x = \rho \int_D y^2 dA = \rho \int_C \langle -y^3/3, 0 \rangle \cdot d\vec{r} = -\rho/3 \int_C y^3 \cdot dx \]

17.5 # 3
\[ \nabla \cdot \vec{F} = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \cdot \langle 1, x + yz, xy - \sqrt{z} \rangle = 0 + z - \frac{1}{2\sqrt{z}} \]
\[ \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & x + yz & xy - \sqrt{z} \end{vmatrix} = \hat{i}(x - y) - \hat{j}(y - 0) + \hat{k}(1 - 0) \]
\[ \nabla \times \vec{F} = \langle x - y, -y, 1 \rangle \]

17.5 # 12
a) Meaningless, since curl operates on vectors.
b) Vector field
c) Scalar field
d) Vector field
e) Meaningless, since gradient operates on scalar fields
f) Vector field
g) Scalar field
h) Meaningless, since divergence operates on vector fields
i) Vector field
j) Meaningless, since divergence operates on vector fields, but the first divergence would return a scalar field
k) Meaningless. The cross product works on vectors, but \( \text{div}(\vec{F}) \) is a scalar.
l) Scalar field. Actually, 0, but that’s a scalar.

17.5 # 13
If \( \nabla \times \vec{F} = \vec{0} \), then \( \vec{F} \) is a conservative field. The cross product is \( \langle (x - x), (-y + y), (z - z) \rangle = \vec{0} \). A good potential function is \( f(x, y, z) = xyz \).