Homework 1

Section 11.1, pg. 692: 8, 24, 43.
Section 11.2, pg. 702: 10, 20 (no graph required), 32, 40.
Section 11.3, pg. 713: 14, 20, 54, 81.
Section 11.4, pg. 719: 6, 35, 46.

Solutions

11.1: #8
a) Solve $x = 1 + 3t$ for $t$, $t = \frac{x-1}{3}$, and plug into $y = 2 - t^2$. You get the cartesian equation $y = \frac{17}{9} - \frac{x^2}{9} + \frac{2x}{9}$.
  b) First generate a table of values:

<table>
<thead>
<tr>
<th>t</th>
<th>x</th>
<th>y</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-5</td>
<td>-2</td>
</tr>
<tr>
<td>-1</td>
<td>-2</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>7</td>
<td>-2</td>
</tr>
</tbody>
</table>

Then plot the values (see figure 1).

Figure 1: Parametric plot for 11.1 #8.

11.1: #24
a - III : x goes from 1 to 2
b - I : y is twice as fast as x

c - IV : y is always positive


d - II : y is almost constant for a span, and x is as well.

11.1: #43

a) This is easiest if you first plot all the functions against t (see fig 2).

Figure 2: Graph of parametric equations vs t for 11.1 #43. Dashed = x, thick is for the first particle, etc..

Now we transfer this information to the to the cartesian plane for the plot of the particle trajectories (see fig 3).

Figure 3: Parametric plot for 11.1 #43. There are two intersections, which are potential collision points.
b) We see two spots where the trajectories cross, but these are only collisions if the x coordinates intersect at the same time the y coordinates do. Looking again at the first plot, we see that this happens only for $t = \frac{3\pi}{2}$.

c) Now there are no collisions. Replot the parametric equations to verify.
11.2: #10
\[
\frac{dx}{dt} = \cos(t)
\]
\[
\frac{dy}{dt} = \cos(t + \sin(t)) \cdot (1 + \cos(t))
\]
\[
\frac{dy}{dx} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}} = \frac{\cos(t+\sin(t)) \cdot (1+\cos(t))}{\cos(t)}
\]

Now find the values of t for which x = 0 and y = 0:
First \( x = \sin(t) = 0 \)
implies \( t = 0 \) and \( \pi \) in one cycle.

As for \( y = \sin(t + \sin(t)) = 0 \), it is hard to find the root, so we test the values from above: 0 and \( \pi \), and we find both values work. So \( t = 0, \pi \) in one cycle. Therefore, \((x, y) = (0, 0)\) when \( t = 0, \pi \). Plug these values into the equation for the tangent and get \( \frac{dy}{dx} = 2 \) when \( t = 0 \), and \( \frac{dy}{dx} = 0 \) when \( t = \pi \). This can be clearly seen from Fig. 4.

![Figure 4: Graph of parametric curve \((\sin(t), \sin(t + \sin(t)))\) for 11.2 #10.](image)

11.2: #20
\[
\frac{dx}{d\theta} = -3\sin\theta = 0 \text{ implies } 3\theta = 0, \pi \text{ in one cycle, so } 3\theta = k\pi \text{ for any integer } k, \text{ so } \theta = \frac{k\pi}{3}.
\]
Therefore in one cycle, \( \theta = 0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3 \), six values in one cycle.

\[
\frac{dy}{d\theta} = 2\cos\theta = 0 \text{ if } \theta = \pi/2, 3\pi/2.
\]

Notice the roots for \( \frac{dx}{d\theta} \) and \( \frac{dy}{d\theta} \) do not overlap.
When the tangent is horizontal, \( \frac{dy}{dx} = 0 \), i.e. when \( \frac{dy}{d\theta} = 0 \), but \( \frac{dx}{d\theta} \neq 0 \), then we get
\( \theta = \pi/2, 3\pi/2 \) in one cycle.

When it is vertical, \( \frac{dy}{dx} \) goes to infinity, i.e. when \( \frac{dx}{d\theta} = 0 \), but \( \frac{dy}{d\theta} \neq 0 \), we get \( \theta = 0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3 \) in one cycle.

Figure 5: Graph of parametric curve \((\cos(3t), 2\sin(t))\) for 11.2 #20.

11.2: #40
Set \( t + 1/t = 2.5 \) to obtain \( t_1 = 0.5 \) and \( t_2 = 2 \), plug them into \( x = t - 1/t \) to get \( x_1 = -1.5 \) and \( x_2 = 1.5 \), from the picture, the area bounded can be calculated by:

\[
A = 2.5 \times (1.5 - (-1.5)) - \int_{0.5}^{2} (t + 1/t)(1 + 1/t^2)dt = 7.5 - 6.522 = 0.977
\]

Figure 6: Graph of parametric curve for 11.2 #32.

11.2: #40
\[
\begin{align*}
\frac{dx}{dt} &= 1/t \\
\frac{dy}{dt} &= \frac{1}{2\sqrt{1+t}} \\
L &= \int_{a}^{b} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt
\end{align*}
\]
\[ L = \int_1^5 \sqrt{1/t^2 + 1/4} \frac{dt}{t} \]

**11.3: #14**

Start with the definition of distance in Cartesian coordinates:

\[ D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \]

Make the substitutions for \( x = r \cdot \cos \theta \) and \( y = r \cdot \sin \theta \) and distribute.

\[ D = \sqrt{(r_1 \cos \theta_1 - r_2 \cos \theta_2)^2 + (r_1 \sin \theta_1 - r_2 \sin \theta_2)^2} \]

Finally, use the sum-angle identity \( \cos(\theta_1 - \theta_2) = \cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \).

\[ D = \sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos(\theta_1 - \theta_2)} \]

**11.3: #20**

\[ r = \tan \theta \sec \theta \]

Multiply by \( \cos \theta \).

\[ r \cos \theta = \tan \theta \]

\[ x = y/x \]

\[ y = x^2 \]

**11.3: #54**

(a): VI, since the period of \( \sin(\theta/2) \) is \( 4\pi \), which needs the initial ray rotate two complete circle.

(b): III, since the period of \( \sin(\theta/4) \) is \( 8\pi \), which needs the initial ray rotate four complete circle.

(c): IV, 3 indicates three pieces and \( r = 1 \) when \( \theta = 0 \).

(d): V, \( -\theta \) will not change the function so symmetric about x-axis and bigger and bigger \( r \) with the increase of \( \theta \).

(e): II, 5 indicates the five pieces.
(f): I, r changes from infinite to smaller and smaller number as θ increase from 0.

### 11.3: #81

The slope of the tangent is given by:

\[
\begin{align*}
\tan \phi &= \frac{dy}{dx} = \frac{\frac{dr}{d\theta} \sin \theta + r \cos \theta}{\frac{dr}{d\theta} \cos \theta - r \sin \theta} \\
\tan \phi &= \frac{dy}{dx} = \frac{\sin \theta + r' \cos \theta}{\cos \theta - r' \sin \theta} \\
\tan \phi &= \frac{dy}{dx} = \frac{\tan \theta + r'/r'}{1 - r/r' \tan \theta}
\end{align*}
\]

using the notation \( r' = \frac{dr}{d\theta} \).

Since \( \psi = \phi - \theta \), \( \tan \psi = \tan(\phi - \theta) \),

\[
\tan \psi = \frac{\tan \phi - \tan \theta}{1 + \tan \phi \cdot \tan \theta}
\]

\[
\tan \psi = \frac{\frac{\tan \theta + r'/r'}{1 - r/r' \tan \theta} - \tan \theta}{1 + \tan \theta \frac{\tan \theta + r'/r'}{1 - r/r' \tan \theta}}
\]

Consider just the numerator: combine the two terms by finding a common denominator. Do this for the bottom of the fraction as well.

\[
\tan \psi = \frac{\tan \theta + r'/r' - \tan \theta - r/r' \tan^2 \theta}{1 - r/r' \tan \theta + \tan^2 \theta + r/r' \tan \theta}
\]

\[
= \frac{r/r'}{1 + \tan^2 \theta}
\]

\[
= \frac{r}{r'^2}
\]
11.4: #6

\[ A = \int_{\pi/2}^{\pi} (1/2)(1 + \sin \theta)^2 d\theta = 2.178 \]

11.4: #8

\[ r = \sin(4\theta) \]

\[ A = \int_{0}^{\pi/2} \frac{1}{2} r^2 d\theta \]
\[ = \int_{0}^{\pi/4} \frac{1}{2} \sin^2(4\theta) d\theta \]
\[ = \frac{1}{2} \left[ \theta/2 - \frac{1}{16} \sin(8\theta) \right]_0^{\pi/4} \]
\[ = \pi/16 \]

11.4: #35

\[ r = 1/2 + \cos \theta \]

The first step is to find the endpoints for the two regions; that is, solve \( r = 0 \).

\[ \cos \theta = -1/2 \]
\[ \theta = \pm \frac{2\pi}{3} + 2n\pi \]

By examination, we find the larger region is given by \(-\frac{2\pi}{3} \leq \theta \leq \frac{2\pi}{3}\), and the smaller region by \(\frac{2\pi}{3} \leq \theta \leq \frac{4\pi}{3}\).

Inner Region

\[ A_{\text{inner}} = \int_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \frac{1}{2}(1/2 + \cos \theta)^2 d\theta \]
\[ = 1/8 \left[ 3\theta + 4\sin \theta + \sin 2\theta \right]_{\frac{2\pi}{3}}^{\frac{4\pi}{3}} \]
\[ = 1/8(2\pi - 3\sqrt{3}) \]

Outer Region

\[ A_{\text{outer}} = \int_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} \frac{1}{2}(1/2 + \cos \theta)^2 d\theta \]
\[ = 1/8 \left[ 3\theta + 4\sin \theta + \sin 2\theta \right]_{-\frac{2\pi}{3}}^{\frac{2\pi}{3}} \]
\[ = 1/8(4\pi + 3\sqrt{3}) \]

Thus the area in between is the difference \( A = A_{\text{outer}} - A_{\text{inner}} = 1/8(2\pi + 6\sqrt{3}) \).

11.4: #46

\[ r = e^{2\theta} \]
\[ \frac{dr}{d\theta} = 2e^{2\theta} \]
\[ L = \int_{0}^{2\pi} \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} d\theta \]
\[ S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt \]

Now make the substitutions into polar coordinates: \( x = r\cos\theta \) and \( y = r\sin\theta \). Note that \( t \) is a dummy variable, we could have just called it \( \theta \). Also note that:

\[
\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta
\]

\[
\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta
\]

Let’s look at the quantity \((\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2\).

\[
(\frac{dy}{d\theta})^2 = (\frac{dr}{d\theta})^2\sin^2\theta + r^2\cos^2\theta + 2r\frac{dr}{d\theta}\cos\theta\sin\theta
\]

\[
(\frac{dy}{d\theta})^2 = (\frac{dr}{d\theta})^2\cos^2\theta + r^2\sin^2\theta - 2r\frac{dr}{d\theta}\cos\theta\sin\theta
\]

\[
(\frac{dy}{d\theta})^2 + (\frac{dy}{d\theta})^2 = (\frac{dr}{d\theta})^2(\cos^2\theta + \sin^2\theta) + r^2(\cos^2\theta + \sin^2\theta)
\]

\[
(\frac{dy}{d\theta})^2 + (\frac{dy}{d\theta})^2 = (\frac{dr}{d\theta})^2 + r^2
\]

Plug in this, and \( y = r\sin\theta \) to get

\[ S = \int_a^b 2\pi r\sin\theta \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} \, d\theta \]

b. Now calculate for \( r = \sqrt{\cos^2\theta} \).

\[
\frac{dr}{d\theta} = \frac{-\sin\theta}{\sqrt{\cos^2\theta}}
\]

\[
S = \int_0^{\pi/4} 2\pi \sqrt{\cos(2\theta)\sin(\theta)} \sqrt{\frac{\sin^2(2\theta)}{\cos^2\theta} + \cos^2\theta} \, d\theta
\]

\[
= \int_0^{\pi/4} 2\pi \sqrt{\cos(2\theta)\sin(\theta)} \sqrt{\frac{\sin^2(2\theta) + \cos^2(2\theta)}{\cos^2\theta}} \, d\theta
\]

\[
= \int_0^{\pi/4} 2\pi \sqrt{\cos(2\theta)\sin(\theta)} \sqrt{\frac{1}{\cos^2\theta}} \, d\theta
\]

\[
= \int_0^{\pi/4} 2\pi \sin(\theta) \, d\theta
\]

\[ S = -2\pi \left[ \cos\left(\frac{\pi}{4}\right) - \cos(0) \right] = 2\pi(1 - 1/\sqrt{2}) \]