Asymptotics of Approximation by Bivariate Linear Splines

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Abstract

An early work in the multivariate case of analysis of adaptive mesh generation was [Nadler 1985]. The setting is the approximation of a smooth bivariate function with polygonal domain by piecewise linear functions that are linear on each triangle in a triangulation of the domain, and the asymptotics as the number of triangles goes to infinity are considered. An asymptotic error estimate was obtained for best $L_2$ approximation in this setting, and used to characterize such an asymptotically optimal sequence of triangulations.

In this talk, the above results are reviewed and extended to the more useful cases of continuous linear (approximating) splines and interpolating (necessarily continuous) linear splines.
Outline

• Background
• Basics of the problem
• Triangle analysis
  • Quadratics
    • Approximation error
    • Best triangle shape
  • General smooth functions
• Triangulation analysis
  • Approximation error estimate
  • Optimal triangulation characterization
  • Algorithm remarks
• Approximation by continuous functions
Background


Earlier work:

1-D optimal partitioning:
- de Boor 1972
- Burchard 1974
- McClure 1975
- Barrow & Smith 1978

2-D analysis techniques:
- Fejes Tóth 1959
- McClure 1976

Some subsequent work:
- D’Azevedo 1989+
- Rippa et al 1990+
- Field 1991+
- Elber 1996
- Berzins 1998+
- Garland et al 1999+
- Bertram et al 2000+
- Cao 2005+
- Babenko 2006+
...
Setting

$Q$ a polygonal domain in $\mathbb{R}^2$, e.g., $[0, 1] \times [0, 1]$

$\triangle_n$: triangulation of $Q$ with $n$ triangles $T_i$

piecewise linear functions on $\triangle_n$:
$S_1(\triangle_n) := \{ f : Q \to \mathbb{R} : f|_{T_i} \in P_1 \}$

$S_1^0(\triangle_n) := S_1(\triangle_n) \cap C^0(Q)$, continuous piecewise linear functions

$u \in C^3(Q)$: to be approximated by $S_1^0(\triangle_n)$

error of best $L_2$ approximation of $u$ on a given triangulation:
$\mathcal{E}(u, \triangle_n) := \text{dist}_{L_2}(u, S_1(\triangle_n))$
Objectives

Objective. Minimize $\mathcal{E}(u, \Delta_n)$ over all triangulations $\Delta_n$

Definition. A triangulation $\Delta_n^*$ is called optimal if

$$\mathcal{E}(u, \Delta_n^*) = \inf_{\Delta_n} \mathcal{E}(u, \Delta_n)$$

Theorem. $\Delta_n^*$ exists.

Geometric characterization of optimal triangulations $\Delta_n^*$ as $n \to \infty$

Based upon asymptotic estimate for $\mathcal{E}(u, \Delta_n^*)$ as $n \to \infty$
Approximation error of a quadratic function on a triangle

\( x_i \) : vertices of \( T \)

\( z_i := x_k - x_j, (i, j, k) \) cyclic, i.e., vectors along sides of \( T \)

\( d : d_i := z_i^T H z_i, i = 1, 2, 3 \)

**Theorem.** The error of best \( L_2 \) approximation by linear functions of a quadratic function with Hessian \( H \) on a triangle of area \( A \) is given by

\[
\mathcal{E}^2 = A d^T P d, \quad \text{with} \quad P := \frac{1}{3600} \begin{bmatrix}
7 & -1 & -1 \\
-1 & 7 & -1 \\
-1 & -1 & 7
\end{bmatrix}
\]
Approximation error of a quadratic function on a triangle

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Equivalently,

\[
E^2 = A^3 v^T R v, \text{ with } v := (u_{xx}, u_{xy}, u_{yy})^T, R := \frac{1}{k^2} Q^T P Q,
\]

with size shape

\[
Q := \begin{bmatrix}
z_{1,1}^2 & 2z_{1,1}z_{1,2} & z_{1,2}^2 \\
z_{2,1}^2 & 2z_{2,1}z_{2,2} & z_{2,2}^2 \\
z_{3,1}^2 & 2z_{3,1}z_{3,2} & z_{3,2}^2
\end{bmatrix}
\]

a form that will be used in subsequent analysis...
Approximation error of a quadratic function on a triangle

\[ x_i : \text{vertices of } T \]
\[ z_i := x_k - x_j, (i, j, k) \text{ cyclic, i.e., vectors along sides of } T \]
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\]

For *interpolation* at the vertices of \( T \), the \( L_2 \) error is of the same form:

\[
\mathcal{E}^2_I = A d^T P_I d, \text{ with } P_I := \frac{1}{180} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}
\]

where \( P \) and \( P_I \) have the same eigenspaces.
Best $L_2$ approximation vs. interpolation at triangle vertices

\[ E^2 = A d^T P d, \text{ with } P := \frac{1}{3600} \begin{bmatrix} 7 & -1 & -1 \\ -1 & 7 & -1 \\ -1 & -1 & 7 \end{bmatrix} \]

\[ E^2_I = A d^T P_I d, \text{ with } P_I := \frac{1}{180} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \]

Eigenspaces of $P$ and $P_I$:
\[ D_1 : d_1 = d_2 = d_3 \]
\[ D_2 : d_1 + d_2 + d_3 = 0 \]

Eigenvectors of $P, P_I$: \((\frac{1}{720}, \frac{1}{450}), (\frac{1}{45}, \frac{1}{180})\)
Best triangle shape for approximation of a quadratic

Minimize $d^T P d$ for constant triangle area $\mathcal{A}$
Best triangle shape for approximation of a quadratic

Minimize $d^T P d$ for constant triangle area $A$:

subject to $d^T S d = A^2 \det H$, \quad $S := \frac{1}{4} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$S$ shares the eigenspaces $\mathcal{D}_1, \mathcal{D}_2$ of $P$ and $P_I$. 
Best triangle shape for approximation of a quadratic

Minimize $d^T P d$ for constant triangle area $A$:

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$S$ shares the eigenspaces $D_1, D_2$ of $P$ and $P_I$.

$$\begin{array}{lcr}
\text{det } H > 0 & \quad & \text{det } H < 0 \\
\hline
d \in D_1 & \quad & d \in D_2 \\
E^2 = \frac{1}{180} A^3 \text{ det } H & \quad & E^2 = \frac{1}{225} A^3 |\text{det } H| \\
z_i^T H z_i = \frac{4}{\sqrt{3}} A (\text{det } H)^{1/2} \quad \forall i & \quad & z_i^T |H| z_i = \frac{4}{\sqrt{3}} A |\text{det } H|^{1/2} \quad \forall i \\
\text{where } |H| \text{ is } H \text{ with all eigenvalues replaced by their absolute values} & \quad & \\
+ \text{ stretch along axes of 0 curvature:} & \quad & \\
\end{array}$$
Best triangle shape for approximation of a quadratic continued

**Theorem** (Optimal triangle shape). The triangles $T^*$ of area $\mathcal{A}$ for which the error of best $L_2$ approximation of a quadratic function with Hessian $H$ by a linear function is minimized, are

$$T^* = \begin{cases} (H^{1/2})^{-1} T_{eq} & \text{with } E^2 = \begin{cases} \frac{1}{180} \mathcal{A}^3 \det H & \det H > 0 \\ \frac{1}{225} \mathcal{A}^3 |\det H| & \det H < 0 \end{cases} \\ \mathcal{V}_p(|H|^{1/2})^{-1} T_{eq} \end{cases}$$

with matrix operators $\{1/2\} := M$ such that $M^T M = \cdot$ and $|\cdot|$ : replaces all eigenvalues by their absolute values

and

where $T_{eq}$ is an equilaterial triangle in any orientation

$\mathcal{V}_p$ is a matrix stretching by $\{p, 1/p\}$ in the 2 directions of curvature 0
Theorem (Optimal triangle shape). The triangles $T^*$ of area $A$ for which the error of best $L_2$ approximation of a quadratic function with Hessian $H$ by a linear function is minimized, are

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with $E^2 = \begin{cases} \frac{1}{180}A^3 \det H & \det H > 0 \\ \frac{1}{225}A^3|\det H| & \det H < 0 \end{cases}$

where $T_{eq}$ is an equilateral triangle in any orientation

$\mathcal{V}_p$ is a matrix stretching by $\{p, 1/p\}$ in the 2 directions of curvature 0

And similarly for interpolation at the vertices of a triangle:

$$E_I^2 = \begin{cases} \frac{8}{90}A^3 \det H & \det H > 0 \\ \frac{1}{90}A^3|\det H| & \det H < 0 \end{cases}$$
Approximation of a smooth function on a triangle

Recall for a quadratic, $\mathcal{E}^2 = A^3 \, v^T R v$, $v = v(H)$, $R$ a measure of triangle shape.

Extend this to $u \in C^3$
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Extend this to $u \in C^3$

$T_h := x_0 + h(T - x_0)$, with area $A_h = h^2 A$,

Expand $u$ in a Taylor series at $x_0 \in T$...

**Theorem.** For any point $x_0$ in a triangle $T$ of area $A$, the error of best $L_2$ approximation of $u \in C^3(Q)$ by a linear function on triangle $T_h$ is given by

$$\mathcal{E}^2(T_h) = A_h^3 v^T(x_0) R v(x_0) (1 + O(h))$$
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Recall for a quadratic, $E^2 = A^3 v^T R v$, $v = v(H)$, $R$ a measure of triangle shape.

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$$J(x) := v^T(x) R v(x) = \lim_{h \to 0} \frac{\mathcal{E}^2(T_h)}{A^3_h}$$

acts as an “error density”
From the optimal shape analysis define

\[ J^*(x) := \begin{cases} 
\frac{1}{180} \det H(x) & \det H(x) > 0 \\
\frac{1}{225} |\det H(x)| & \det H(x) < 0 
\end{cases} \]

\( J \geq J^* \), with equality iff the triangle conforms to \( H \), i.e., is optimal in the sense of the Optimal triangle shape theorem.

**Definition.** A triangle is said to be well-shaped if it conforms to \( H \) at some point within it.
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**Definition.** A triangle is said to be well-shaped if it conforms to \( H \) at some point within it.

Similarly for interpolation at the vertices of the triangle,

\[ J^*_I(x) := \begin{cases} \frac{8}{90} \det H(x) & \det H(x) > 0 \\ \frac{1}{90} |\det H(x)| & \det H(x) < 0 \end{cases} \]
Regular triangulation with $n$ triangles, obtained by linear transformation by $(H^{1/2})^{-1}$ of a grid of equilateral triangles

$$E^2 = \sum E_i^2 \sim \sum A_i^3 J$$

from which easily follows

$$\lim_{n \to \infty} n^2 E_n^2 = A^2 \int_Q J$$
Optimal triangulation - introduction

Regular triangulation with \( n \) triangles, obtained by linear transformation by \((H^{1/2})^{-1}\) of a grid of equilateral triangles

\[
\mathcal{E}^2 = \sum \mathcal{E}_i^2 \sim \sum A_i^3 J
\]

from which easily follows

\[
\lim_{n \to \infty} n^2 \mathcal{E}_n^2 = A^2 \int_Q J
\]

1-D analogue: approximation on an interval

equal partition: \( \lim_{n \to \infty} n^4 \mathcal{E}_n^2 = \frac{1}{720} \text{Length}^4 \int u''^2 \)

optimal partition: \( \lim_{n \to \infty} n^4 \mathcal{E}_n^*^2 = \frac{1}{720} \left( \int u''^\frac{2}{3} \right)^5 \) [McClure 1975]
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optimal partition: \( \lim_{n \to \infty} n^4 \mathcal{E}_n^{*2} = \frac{1}{720} \left( \int u''^2 \right)^5 \) [McClure 1975]

By analogy with 1-D case, expect 2-D estimate of the form

\[
\lim_{n \to \infty} n^2 \mathcal{E}_n^{*2} = \left( \int_Q J^{*p} \right)^q
\]

Must restrict attention to \( u \) with \( \det H \) bounded away from 0 on \( Q \).
Estimate suggested by series of lower bounds on a quantity asymptotically equal to $E^2$

$$\sum_{i=1}^{n} J_i(\xi_i)A_i^3 \geq \sum_{i=1}^{n} J^*(\xi_i)A_i^3 \geq \frac{1}{n^2}\left(\sum_{i=1}^{n} J^*(\xi_i)^{\frac{1}{3}}A_i\right)^3$$
Optimal triangulation - developing the estimate

Estimate suggested by series of lower bounds on a quantity asymptotically equal to $E^2$

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Equality is achieved in the first inequality iff each triangle is well-shaped, and in the second iff $A_i \propto J^*(\xi_i)^{-\frac{1}{3}}$
Optimal triangulation - developing the estimate

Estimate suggested by series of lower bounds on a quantity asymptotically equal to $\mathcal{E}^2$

$$\sum_{i=1}^{n} J_i(\xi_i) A_i^3 \geq \sum_{i=1}^{n} J^*(\xi_i) A_i^3 \geq \frac{1}{n^2} \left( \sum_{i=1}^{n} J^*(\xi_i)^{\frac{1}{3}} A_i \right)^3$$

Equality is achieved in the first inequality iff each triangle is *well-shaped*, and in the second iff $A_i \propto J^*(\xi_i)^{-\frac{1}{3}}$

$\mathcal{E}(\triangle_n) \to 0$ implies $A_i \to 0$ *uniformly*, from which follows

**Lemma.** The sequence $\mathcal{E}_n^* := \mathcal{E}(u, \triangle_n^*)$ satisfies

$$\liminf_{n \to \infty} n^2 \mathcal{E}_n^* \geq \left( \int_{Q} J^*(x)^{\frac{1}{3}} \right)^3$$
Construct a sequence of triangulations for which
\[ \limsup_{n \to \infty} n^2 \mathcal{E}_n^2 \leq \left( \int_Q J^*(x)^{\frac{1}{3}} \right)^3 \]
to establish our main result:

**Theorem** (Optimal triangulation estimate). The sequence \( \mathcal{E}_n^* \) of optimal errors for best \( L_2 \) approximation by \( S_1(\triangle_n) \) satisfies
\[
\lim_{n \to \infty} n^2 \mathcal{E}_n^2 = \left( \int_Q J^*(x)^{\frac{1}{3}} \right)^3 \quad (\star)
\]
Optimal triangulation - the estimate

Construct a sequence of triangulations for which
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where, again,

\[ J^*(x) := \begin{cases} \frac{1}{180} \det H(x) & \det H(x) > 0 \\ \frac{1}{225} |\det H(x)| & \det H(x) < 0 \end{cases} \]
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\end{cases} \]

Similarly for interpolation at the vertices of triangles, with

\[ J_1^*(x) := \begin{cases} 
\frac{8}{90} \det H(x) & \det H(x) > 0 \\
\frac{1}{90} |\det H(x)| & \det H(x) < 0 
\end{cases} \]
Characterization of optimal triangulations

• The *Optimal triangulation estimate* says that the lower bounds in the inequalities are attainable, in an asymptotic sense.

• The conditions for equality in these two inequalities essentially provide geometric characterizations of optimal triangulations.

• In the following
  • $\{\triangle^*_n\}$: sequence of triangulations satisfying $(\ast)$
  • $T_i$: triangle in $\triangle^*_n$
  • $A_i$: area of $T_i$
Characterization of optimal triangulations

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  - $\{\triangle_n^*\}$: sequence of triangulations satisfying (*)
  - $T_i$: triangle in $\triangle_n^*$
  - $A_i$: area of $T_i$

Result: $A_i \propto J^*(x)^{-\frac{1}{3}}$ asymptotically. Formally...

**Theorem** (Triangle size characterization). Let $R$ be an open subset of $Q$. Then the fractional number of triangles contained in $R$ approaches the following limit as $n \to \infty$

$$\frac{\int_R J^*(x)^{\frac{1}{3}}}{\int_Q J^*(x)^{\frac{1}{3}}}$$
Characterization of optimal triangulations continued

- $\{\Delta^*_n\}$: sequence of triangulations satisfying (*)
- $T_i$: triangle in $\Delta^*_n$
- $A_i$: area of $T_i$
- $x_i$: point in $T_i$. 
Characterization of optimal triangulations continued

- \( \{ \triangle^*_n \} \): sequence of triangulations satisfying (*)
- \( T_i \): triangle in \( \triangle^*_n \)
- \( A_i \): area of \( T_i \)
- \( x_i \): point in \( T_i \).

Result: \( \mathcal{E}^2(u, T_i) \sim A_i^3 J^*(x_i) \). Formally...

**Theorem (Triangle shape characterization).** Let
\( r_i := \mathcal{E}^2(u, T_i)/A_i^3 J^*(x_i) \), and let \( m_n(t) \) be the total area of triangles in \( \triangle^*_n \) with \( r < t \). Then for any \( t > 1 \), \( m_n(t) \to A \), the total area of \( Q \).
Characterization of optimal triangulations continued

- $\{\triangle^*_n\}$: sequence of triangulations satisfying $(\ast)$
- $T_i$: triangle in $\triangle^*_n$
- $A_i$: area of $T_i$
- $x_i$: point in $T_i$.

Result: $\mathcal{E}^2(u, T_i) \sim A_i^3 J^*(x_i)$. Formally...

**Theorem** (Triangle shape characterization). Let

$$r_i := \mathcal{E}^2(u, T_i) / A_i^3 J^*(x_i),$$

and let $m_n(t)$ be the total area of triangles in $\triangle_n$ with $r < t$. Then for any $t > 1$, $m_n(t) \to A$, the total area of $Q$.

The above shape characterization, combined with the previous size characterization, $A_i \propto J^*(x)^{-\frac{1}{3}}$ asymptotically, says that $\mathcal{E}$ is asymptotically *balanced* over the triangles in $\triangle^*_n$. 
Characterization \implies triangulation algorithm

One idea: (used in proof of *Optimal triangulation estimate*):
Coarsely partition $Q$ into subregions $Q_i$, and triangulate as much of $Q_i$ as possible by a regular grid of triangles that conform to $H(x_i), x_i \in Q_i$, “fixed up” near the boundaries of $Q_i$
Better idea:

*Continuous* variation in triangle size and shape to reflect $H(x)$. 
Characterization $\implies$ triangulation algorithm continued

Better idea:

Continuous variation in triangle size and shape to reflect $H(x)$. 

• Generalize distribution function method [Sacks & Ylvisaker 1970] of 1-D, in which a function determines a partition by applying its inverse to even partition.

• In 2-D, from the size & shape characterizations, $\left| \det H \right|^{-\frac{1}{12}} \left| H \right|^{\frac{1}{2}}$ acts like a density function; its inverse applied to a fixed small equilateral approximates a triangle of $\Delta^*_n$.

• May be possible to transform a grid of equilateral triangles to satisfy this density function

Extension to approximation by continuous functions

• In 1-D, best $L^2$ approximation by piecewise linear functions with optimal knots is automatically continuous [Barrow & Smith 1978].

• In 2-D, it turns out that best $L^2$ approximation of quadratics by piecewise linear functions on an optimal triangulation, i.e., an equilateral grid transformed by $(|H|^{1/2})^{-1}$, is automatically continuous.
Extension to approximation by continuous functions

• In 1-D, best $L^2$ approximation by piecewise linear functions with optimal knots is automatically continuous [Barrow & Smith 1978].

• In 2-D, it turns out that best $L^2$ approximation of quadratics by piecewise linear functions on an optimal triangulation, i.e., an equilateral grid transformed by $(|H|^{1/2})^{-1}$, is automatically continuous.

• Moreover, the same holds for the sub-optimal triangulations that are transformations by $(|H|^{1/2})^{-1}$ of the following three regular ones.

• These four triangular triangulations are the ones that fill the plane with their reflections.
For general smooth functions $u$, the best $L^2$ approximant by piecewise linear functions on a sequence of optimal triangulations $\Delta^*_n$ are asymptotically “nearly continuous”.

Consider the continuous piecewise linear approximant obtained by selecting a value at each vertex in the range of the values of the not-necessarily continuous approximant from the triangles sharing the vertex.

The continuous function so constructed satisfies $(\ast)$. Hence

**Theorem.** The main results for *Optimal triangulation estimate* and *Triangle size & shape characterization* hold for approximation by *continuous* piecewise linear functions.
Summary

• Review old results
  • approximation by linear functions on triangles
  • asymptotic results for approximation by piecewise linear functions on triangulations

• New results
  • optimal interpolation by linear functions on triangles
  • asymptotic results for interpolation by piecewise linear functions on triangulations
  • asymptotic results for approximation by continuous piecewise linear functions on triangulations