(1) Consider \( f(z) = e^{-2\pi i z} \). 

\( f(z) \) is obviously an entire function. We also have that \( |f(z)| \leq e^{-2\pi} e^{2\pi |z|} < 1 \) on \( D \)

\[ \Rightarrow f : D \rightarrow D. \] Clearly, we have

\[ f(-\frac{1}{2}) = e^{-2\pi i (-\frac{1}{2})} = e^{-2\pi i (-\frac{1}{2})} = e^{2\pi i / 2} = f(\frac{1}{2}). \] Also, \( f'(z) =

\[ e^{2\pi i z} = 0 \text{ on } D. \]
(2) $f$ is analytic on $0.9|D|$ and continuous on its boundary. By the Maximum Modulus Principle, on $0.9|D|$, $f$ attains its max on $20.9|D|$. Thus, on $0.9|D|$, $|f(z)| \leq 1 - (0.9)^2 + (0.9)^{1000} = 0.19 + (0.9)^{1000} < 0.2 \Rightarrow |f(0)| \leq 0.2$. \(//\)
(4) By the Reimann Mapping Thm, we have that \( \mathbb{D}_2 \) is conformally equivalent to \( \mathbb{D} \). Thus, \( \exists f : \mathbb{D}_1 \rightarrow \mathbb{D} \) iff
\( \exists f : \mathbb{D}_1 \rightarrow \mathbb{D}_2 \). Now, suppose \( f : \mathbb{D}_1 \rightarrow \mathbb{D} \). Then \( f \) is bounded \( \Rightarrow \)
All singularities are removable \( \Rightarrow \)
f can be extended to a function \( f : \mathbb{C} \rightarrow \mathbb{D} \). However, Louivilles Thm says that any bounded entire function is constant
\( \Rightarrow \) Such a map does not exist. \( \triangleright \)
(5) We have that \( \sqrt{z^2-1} = z \left(1 - \left(\frac{1}{2}z^2\right)^2\right)^{\frac{1}{2}} \)

\[
= \left( z \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n \left( -\frac{1}{2} z^2 \right)^n \right)^{\frac{1}{2}}
\]

\[
= z \left[1 + \frac{1}{2} \left( -\frac{1}{2} z^2 \right) + \frac{1}{2} \left( -\frac{1}{2} \right)^2 \left( -\frac{1}{2} z^2 \right)^2 + \cdots \right]
\]

\[= z - \frac{1}{2} z^2 - \frac{1}{4} z^3 + \cdots \]

\( \Rightarrow \quad \alpha = 1, \quad \beta = 0, \quad \gamma = -\frac{1}{2}, \quad \delta = 0, \quad \epsilon = -\frac{1}{4} \)

(b) \( \int_{|z|=2} \frac{(5+6z+7z^2)f(z)}{z} \, dz \)

\[
\int_{|z|=2} \frac{(5+6z+7z^2)}{z} \left[ z - \frac{1}{2} z^2 - \frac{1}{4} z^3 + \cdots \right] \, dz
\]

\[
= \int_{|z|=2} \left[ -\frac{5}{2} z^{-1} - \frac{9}{4} z^{-2} + \cdots \right] \, dz
\]

\[
= 2\pi i \left[ -\frac{17}{4} \right]
\]
(1) First we extend \( f \) to a continuous function on \( IR \) by defining 
\[ f(x) = f(a), \quad x \leq a, \]
\[ f(x) = f(b), \quad x \geq b. \]

Now, define 
\[ f_n(x) = n \left[ f(x + \frac{1}{n}) - f(x) \right]. \]

Then each \( f_n \) is measurable so 
\[ a.e. \quad f_n \rightarrow f^+, \quad f^+(x) \quad \text{is a.e. measurable function}, \quad f^+: IR \rightarrow IR. \]

Now, define 
\[ g_n(x) = -n \left[ f(x - \frac{1}{n}) - f(x) \right]. \]
\[ g_n \quad a.e. \rightarrow f^-, \quad f^-(x) \quad \text{is a.e. measurable}, \quad f^-: IR \rightarrow IR. \]

Let 
\[ A = f^+(\infty, \infty), \quad B = f^-(\infty, \infty). \]

Since \( f^+ \) and \( f^- \) are measurable functions, we have that \( A \cap B \) are measurable 
\[ \Rightarrow A \cap B \quad \text{is measurable}. \]
The set of pts for which $f$ is differentiable is equal to
$$(f^+ - f^-)|_{A \cap B}(0)$$ which is the inverse image of a closed set of the difference of two measurable functions (in range $\mathbb{R}$ since restricted to $A \cap B$ so makes sense) $\Rightarrow$ it is measurable.
\[ f_n \to f \text{ is measure } \Rightarrow \exists \text{ a subsequence } \]
\[ \exists f_{n_j} \text{ such that } f_{n_j} \to f \text{ a.e.} \]
\[ \text{But}, \quad \sup_{n_j} f_{n_j} \text{ a.e.} \]
\[ f = \sup_{n_j} f_{n_j} \leq g \Rightarrow f \leq g. \]
4. (a) \( \|Tf\|_1 = \int \int f(x,y) \text{d}y \text{d}x \)

\[ \leq \int \int |f(x,y)| \text{d}y \text{d}x. \]

Since \( \|Tf\|_1 = \|4f\|_1 = 4 \|f\|_1 \leq \infty \)

We can apply Fubini's Theorem, and so we get:

\[ \int \int |f(x,y)| \text{d}y \text{d}x = \int \int |f(x,y)| \text{d}x \text{d}y \]

\[ = 2 \int \int |f(x,y)| \text{d}y = 2 \|f\|_1, \]

So, \( 4 \|f\|_1 = \infty \|f\|_1, \quad \infty \leq 2 \)

\[ \Rightarrow \|f\|_1 = 0 \Rightarrow f = 0 \text{ a.e.} \]
4. (b) \( \int_{x-1}^{x+1} e^{y} \, dy = 4e^{x} \)

\[ \Rightarrow e^{y} \bigg|_{x-1}^{x+1} = 4e^{x} \]

\[ \Rightarrow e^{x+1} - e^{x-1} = 4e^{x} \]

\[ \Rightarrow e^{x} \left[ e^{x} - e^{-x} \right] = 4e^{x} \]

\[ \Rightarrow 2\sinh(r) = 4r \]

\[ \Rightarrow \sinh(r) = 2r \]

Picture is

\[ \Rightarrow \text{intersects at } r = 0, \Gamma_0, -\Gamma_0. \]

Now, \( r = 0 \) does not work (it actually we multiplied by \( r \) so division by 0 goes), \( \Gamma_0 \) + \( -\Gamma_0 \) works.
So, we have that $Ae^{r_0x} + Ae^{-r_0x}$ works, in particular $\cosh(r_0x)$ and $\sinh(r_0x)$ are good fns.