1. (a) Let $C \subseteq A$ be closed, $A \subseteq X$ compact

$\implies C \subseteq A$ compact $\implies$ every closed subset of $A$ is compact. So, $q/A$ takes closed sets to compact sets. But $Y$ is Hausdorff and compact in Hausdorff $\implies$ closed. So $q/A$ is closed $\implies$ quotient map.

(b) Let $X = [0,1] \times [0,1]$, $Y = [0,1]$, $q : X \mapsto Y$ be projection. Then $X$, $Y$, $q$ satisfy the hypothesis. Let $B = \overset{\circ}{[\frac{1}{2}, 1]}$. Then $q^{-1}(U)$ where $U = [\frac{1}{2}, 1]$ is open in $B$ but not in $Y$ $\implies$ $q$ not quotient.
2. If there did, then $X(S^n) = k \cdot X(X)$

$$2 = k \cdot X(X) \Rightarrow \frac{2}{k} = X(X). \text{ But } \frac{2}{k} \in (0, 1)$$

$$\Rightarrow \Rightarrow \Rightarrow$$

3. (a) $d g(x, y, z) = \left< 8x, 2y, 2z \right> = (0, 0, 0) \Rightarrow$

$$x = y = z. \text{ But } 4x^2 + y^2 + z^2 = 1 \Rightarrow x, y, z \text{ not all } 0$$

$$\Rightarrow 1 \text{ is a regular value of } g.$$

(b) Critical points occur whenever

$$(2x, 0, -1) \text{ and } (8x, 2y, 2z) \text{ are linearly dependent}.$$

We can't have $x = y = z = 0 \Rightarrow Z \neq 0$

$$\Rightarrow (2x, 0, -1) = (8x, 2y, 2z)$$

$$\Rightarrow -4xz = 8x, \ y = 0$$

$$\Rightarrow x = 0 \text{ or } z = -z \Rightarrow \text{Critical Points}$$

$$\text{are } M_n \{ (0, 0, z) | z \in \mathbb{R} \} \cup M_n \{ (x, 0, -z) | x \in \mathbb{R} \}$$

But $z^2 = 18 \Rightarrow 4x^2 + 4 = 18 \Rightarrow x^2 = \frac{7}{2}$

Critical Points are $(0, 0, \pm \sqrt{2}) \cup (\pm \sqrt{2}, 0, 0)$
4. \( X = D \cup S^3 \). We use MV and we let \( U = D \), \( V = S^3 \), \( U \cap V = S^1 \).

We have \( 0 \to \tilde{H}_3(S^1) \to \tilde{H}_3(D) \oplus \tilde{H}_3(S^3) \to \tilde{H}_2(X) \oplus \tilde{H}_2(S^1) \to \tilde{H}_2(D) \oplus \tilde{H}_2(S^3) \to \tilde{H}_1(X) \to \tilde{H}_1(S^1) \to \tilde{H}_1(D) \oplus \tilde{H}_1(S^3) \to \tilde{H}_1(X) \to 0 \)

\( \Rightarrow \tilde{H}_3(X) \cong \mathbb{Z} \), \( \tilde{H}_2(X) \cong \mathbb{Z} \), \( \tilde{H}_1(X) \cong 0 \)

\( \Rightarrow H_n(X) = 0 \) if \( n \geq 4 \), \( H_3(X) = H_2(X) = H_0(X) = \mathbb{Z} \)

\( H_1(X) = 0 \). /
Now finitely many of these $V_x$ cover $f^{-1}(y)$, say $V_1, \ldots, V_N$.

Then $\bigcup_{i=1}^{N} V_i$ open $\Rightarrow X - \bigcup_{i=1}^{N} V_i$ closed

$f(X - \bigcup_{i=1}^{N} V_i) = f(X) - \bigcup_{i=1}^{N} f(V_i) = \text{closed in } Y$

$\Rightarrow \bigcup_{i=1}^{N} f(V_i)$ open and by construction $f(\bigcup_{i=1}^{N} C_i) = C$

closed and $U \subseteq C \subseteq U_y \Rightarrow Y$ is locally compact.
Afternoon

1. Define \( g : S^1 \rightarrow S^2 \) by \( g(x) = \frac{d f_x}{\| d f_x \|} \).

That is, for \( x \in S^1 \), \( g \) takes \( x \) to the unit tangent vector at \( f(x) \). This is a smooth map and the origin is in the plane \( C \) s.t. \( \partial \circ f : S^1 \rightarrow C \) has \( d(\partial \circ f)_x \neq 0 \). For \( \forall x \in S^1 \), it does for \( \partial \circ g \). Note that \( g \) is smooth so it misses two points \( \mathbf{v}_1, \mathbf{v}_2 \) of \( S^2 \) since \( \dim(S^1) < \dim(S^2) \). Let \( C \) be the plane orthogonal to \( \mathbf{v}_1 \). This does the trick. \( \Box \)
2. Note that subgroups of $F_5$ of index 3 are in bijective correspondence with 3-fold coverings of $V S^1 = X$. Let $f: Y \rightarrow X$ be a 3-fold covering. Then we have $X(X) = 1 - 5 = -4 \Rightarrow X(Y) = -12 = |H_0(Y)| - |H_1(Y)| \Rightarrow H_1(Y) = \frac{13}{3}$

$\Rightarrow Y \cong V S^1 \Rightarrow$ Only subgroups of $F_5$ of index 3 are those $\pi_1(\bigvee_{\stackrel{2}{i=1}}^{\frac{13}{3}} V S^1) = F_{\frac{13}{3}}$. \"
3. Note $X^p \times 0$, i.e. a pinch torus.

Then any neighborhood of $p$ contains $X$, i.e. a double cone & no nbd of $p \in R^2$. We can also use homology.

$$\begin{align*}
0 & \to \tilde{H}_2(X - \{p\}) \to \tilde{H}_2(X) \to \tilde{H}_2(X, X - \{p\}) \\
\tilde{H}_1(X - \{p\}) & \to \tilde{H}_1(X) \to \tilde{H}_1(X, X - \{p\}) \to 0
\end{align*}$$

$X - \{p\} \cong S^1$, $\tilde{H}_1(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ if manifold,

$\tilde{H}_1(X, X - \{p\}) = 0$

$\Rightarrow$ We have $\ldots \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to 0$

$\Rightarrow \ker b = \text{im } a = \mathbb{Z} \oplus \mathbb{Z}$

$\Rightarrow \tilde{H}_1(X, X - \{p\}) \neq 0 \Rightarrow X$ is not a topological manifold.
1. (a) R. Clear

(b) Comb Space \( X = [0,1] \times \mathbb{R}^3 \cup \{ \frac{1}{n} \times [0,1] \mid n \in \mathbb{Z} > 0 \} \cup \mathbb{R}^3 \times [0,1] \).

It is clear this is path connected ⇒ connected. Not locally path connected. Take any nbhd of \((0,1)\) of radius \(\leq \frac{1}{10}\).

Then the nbhd is a disjoint union of \(\infty\) vertical line pieces.

(c) Take a complete metric space which is not compact. For example we can take \(\mathbb{R}^2\). Let \(d\) be the standard metric on \(\mathbb{R}^2\).

Define \(d'(x,y) = \min \{d(x,y), 1\} \). Then it's bounded.
5. Use Van-Kampen. Then we have that $U \cap V = S^1$, $U = \ast$, $V = \mathbb{R}P^2$.

\[
\begin{array}{c}
\pi_1(S^1) \xrightarrow{g_*} 0 \\
\downarrow \quad \downarrow \\
\pi_1(\mathbb{R}P^2) \quad \pi_1(X)
\end{array}
\]

\[
\Rightarrow \pi_1(X) \cong \pi_1(\mathbb{R}P^2) \cong \mathbb{Z}_2.
\]

since $f_*$ must send any generator to 0 since any nonzero element in $\pi_1(S^1)$ has order $\infty$. //