1. (a) Suppose $X \times Y$ is not connected $\Rightarrow$

$X \times Y = U \sqcup V$ with $U, V$ open. Let $U = U_1 \times U_2$, $V = V_1 \times V_2$. Then $U_1, U_2$ open in $X$, $V_1, V_2$ open in $Y$ since projection maps are open. Note, $U \cap V = \emptyset \Rightarrow U \cap V_1 = \emptyset$ or $U_2 \cap V_2 = \emptyset \Rightarrow$

either $X = U_1 \sqcup V_1$ or $Y = U_2 \sqcup V_2 \Rightarrow$

$\Rightarrow X \times Y$ is connected.

(b) Let $x, y \in \mathbb{R}^\infty$. Let

$\gamma(t) : [0, 1] \rightarrow \mathbb{R}^\infty$ be given by $\gamma(t) = x + (y - x)t$.

This is a path and therefore $\mathbb{R}^\infty$ is path-connected $\Rightarrow$ connected.
2. Suppose \( X \) is a manifold. Clear \( \text{Inbd} \times \mathbb{R}^n \) for any pt. in interior of quadrant under quotient map \( \Rightarrow \) \( n \)-manifold.

Let \( U_0 \) be a nbd around \( 0_{2n} \approx \mathbb{R}^n \). Then we have that \( U_0 \times s^{n-1} \approx \mathbb{R}P^{n-1} \neq S^{n-1} \) for \( n \geq 3 \). \( \Rightarrow \) \( X \) is not a topological manifold for \( n \geq 3 \). Note \( \mathbb{R}/\mathbb{n} = \) \( \rightarrow \) Which is not a topological manifold.

Now, \( \mathbb{R}^2/\mathbb{n} = \begin{array}{c}
A \setminus \text{cone} \approx \mathbb{R}^2 \Rightarrow \text{X is a topological manifold} \\
\end{array} \) 

\( \Leftrightarrow n = 2. \)
3. Let \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \) be defined by

\[
f(x,y,z) = x^2 + 2x + 3y^2 - 6y + z^2 + 4z
\]

\[
= (x+1)^2 - 1 + 3(y^2 - 2y) + (z+2)^2 - 4
\]

\[
= (x+1)^2 - 1 + 3(y-1)^2 - 3 + (z+2)^2 - 4
\]

\[
= (x+1)^2 + 3(y-1)^2 + (z+2)^2 - 8
\]

Let \( g : \mathbb{R}^3 \rightarrow \mathbb{R} \) be given by

\[
g(x,y,z) = (x+1)^2 + 3(y-1)^2 + (z+2)^2.
\]

Want to figure out when \( f^{-1}(c+8) \) is a 2-manifold.

\[
\text{d}g(x,y,z) = (2(x+1), 6(y-1), 2(z+2)) \implies \text{Worry some points are } (-1, 1, -2). \text{ If } c > -8, \text{ then } c+8 \text{ is a regular value } \implies \text{Smooth 2-manifold. If } c < -8 \implies \text{it is empty } \implies \text{Not 2-manifold. If } c = 1 \implies \text{single point} \implies 0\text{-manifold and not 2-manifold.}
\]
4. \( Y \cong T \vee T \vee S \)
\[ \Rightarrow \pi_1(\mathbb{Z} \times \mathbb{Z})* (\mathbb{Z} \times \mathbb{Z})* \mathbb{Z}. \]

5. We will show that \( f \) is a closed mapping.

Let \( K \subset X \) be closed. Let \( y \) be a limit point of \( f(K) \).

Let \( U_y \) be a compact nbhd of \( y \) in \( \mathbb{R}^n \). Then, \( y \) is a limit point of \( f(K) \iff y \) is a limit point of \( f(K) \cap U_y \).

Since \( f \) is a proper map, we have that \( f^{-1}(U_y) \) is compact.

So, \( K \cap f^{-1}(U_y) \) is compact in \( X \).

But, \( f(K \cap f^{-1}(U_y)) = f(K) \cap U_y \) is compact \( \Rightarrow \)

closed \( \Rightarrow y \in f(K) \cap U_y \Rightarrow y \in f(K) \Rightarrow f(K) \) contains all of its limit points \( \Rightarrow f(K) \) is closed. \( \Rightarrow \)

\( f : X \mapsto f(X) \) is a continuous bijection of closed sets \( \Rightarrow \)

homeomorphism. \( \Box \)
1. We can equivalently think of $X$ as a torus $T$ formed by identifying a 2-disk in $T$ with boundary $C$ to $M_4D$ where $M_4D$ is a mobius band with disk glued along boundary and this disk and boundary identified to 2-disk in $T$.

But $M_4D \cong \mathbb{RP}^2$ and a 2-disk contracts to a point $\Rightarrow X \cong T \vee \mathbb{RP}^2 \Rightarrow$

$H_n(X) \cong 0$ for $n \geq 3$, $H_2(X) \cong \mathbb{Z}$,

$H_1(X) \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2$, $H_0(X) \cong \mathbb{Z}$. //
2. Let \( y \in Y \). Then for each \( x \in f^{-1}(y) \), \( \exists U_x \) open \( x \in U_x \), \( f: U_x \rightarrow f(U_x) \) homeomorphism. Thus \( f^{-1}(y) \) must be discrete but \( X \) compact Hausdorff \( \Rightarrow \) finite.

Let \( f^{-1}(y) = \{x_1, x_2, \ldots, x_n\} \). Let \( U_1, \ldots, U_n \) be open sets around these points with \( f: U_i \rightarrow f(U_i) \) homeomorphism.

Let \( V = \cap_{i=1}^{n} f(U_i) \). Then \( f: U_i \cap f^{-1}(V) \rightarrow V \) is a homeomorphism \( \forall U_i \Rightarrow f \) is a cover as Douglas \( f \) is onto. \( f(X) \) compact in Hausdorff space \( \Rightarrow \) closed.

Also open since \( f \) is a local homeomorphism \( \Rightarrow f(X) \) open and closed.

Since \( Y \) connected \( \Rightarrow f(X) = Y \Rightarrow f \) is a covering. \( \Box \)
3. (a) $\pi_1(S^3) = \mathbb{Z} \Rightarrow$ Does not.

(b) $S'$ nontrivially covers itself. For example we have $f: S^1 \to S'$ given by $f(z) = z^3$ is a 3-fold cover,

$\Rightarrow S' \times S'$ nontrivially covers itself.

(c) Since $S'$ nontrivially covers itself $\Rightarrow S^1 \times S^2$ does.

(d) $\chi(\Sigma_2) = 1 - 4 + 1 = -2$

$\Rightarrow -2 = K(-2) \Rightarrow K = 1 \Rightarrow$ cannot nontrivially cover itself. "$
4. First note that there is a homeomorphism
\[ f: \overline{B}_1(0) \rightarrow \overline{B}_1(0) \] where \( \overline{B}_1(0) \) denotes the closed ball of radius 1; that is, the identity on the boundary and which switches the interior points \( x + y \). Now, \( M \) connected manifold \( \Rightarrow \) also locally path-connected \( \Rightarrow \) path connected.

Now, let \( x, y \in M \). Let \( p: [0, 1] \rightarrow M \) be a path from \( x \) to \( y \).

Then \( p([0, 1]) \) is compact in \( M \). Consider the open cover
\[ \mathcal{O} \] of \( p([0, 1]) \) given by \( \mathcal{O} = \{ U_x \mid x \in p([0, 1]) \text{ and } U_x \subset B_1(0) \} \).

Then \( \mathcal{O} \) has a subcover, say \( \{ U_1, \ldots, U_n \} \). After reordering, we may assume \( x \in U_1, y \in U_n, U_i \cap U_{i+1} \neq \emptyset \).

Let \( x_i \in U_i \cap U_{i+1} \). Produce a sequence \( x, x_1, x_2, \ldots, x_{n-1}, y \).

Then we have homeomorphisms \( M \setminus \{x\} \mapsto M \setminus \{x_1\} \mapsto \cdots \mapsto M \setminus \{y\} \mapsto M \setminus \{y\} \).
(a) 

5. \( \implies \) Let \( x, y \in CX \). if \( x, y \in X \times (0,1) \)

Then Hausdorffness is clear since \( X \times (0,1) \) is Hausdorff. Suppose \( x \neq y \), \( x = X \times \delta_0 \).

Then \( y = (x,a) a \neq 0 \). So, we can choose the open set \( X \times [0, \frac{1}{8}a] \) around \( x \) and \( X \times [\frac{1}{4}a, 1] \) around \( y \). \( \leftarrow \) Since \( CX \) is Hausdorff \( \implies \)

there are disjoint neighborhoods \( x \times \delta \frac{1}{2} \) and \( y \times \delta \frac{1}{2} \) for \( x \neq y \).

Then \( q : X \times [0,1] \rightarrow CX \Rightarrow q^{-1}(U), q^{-1}(U) \) disjoint and open in \( X \times [0,1] \). Since projection maps are open \( \implies \) \( \Pi_X(q^{-1}(U)) \) and \( \Pi_X(q^{-1}(U)) \) are disjoint open

nbhds of \( x \) and \( y \) in \( X \).
5. (b) Suppose $X$ is Hausdorff. Then $CX$ is connected since $CX$ is the image of $X$ under a quotient map, and quotient maps are continuous and the continuous image of a connected space is connected. Thus, suppose $CX$ is connected. Actually this statement is false. Let $X = \mathbb{R}$, $X = [0, 13$. Then $X \times [0, 1] = [0, 1] \cup [0, 1]$ and $CX = \langle$ identifies them at a point $\Rightarrow CX$ connected but $X$ is not.