(1) This is false, let \( E = \bigcup_{n=1}^{\infty} (n-2, n+2) \). Then, \( m(E) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 1 \).

E is unbounded \( \Rightarrow \) No finite union of bounded intervals cover E \( \Rightarrow m(F) = +\infty \). 

\( \Rightarrow F \) does not necessarily exist.

(2) Equivalently, we will show that

\[ g(p) = \|f\|_p^p \text{ is a continuous function.} \]

Note, \( g(p) = \int |f|^p = \int |f|^p + \int |f|^p \)

\( \text{if } |f| < 1 \quad \text{if } |f| \geq 1 \)

\[ = g_1(p) + g_2(p). \]

We show that \( g_1 \) is continuous.
For $g_1$, we have that for $|\varepsilon|$ small,

$|f|^{p+\varepsilon}$ is dominated by $|f|^{p}$

$\Rightarrow \lim_{\varepsilon \to 0} g_1(p+\varepsilon) = g_1(p)$

Now, for $|\varepsilon|$ small, for $g_2$ we have

$|f|^{p+\varepsilon}$ is dominated by $|f|^{q}$

$\Rightarrow \lim_{\varepsilon \to 0} g_2(p+\varepsilon) = g_2(p) \Rightarrow g_1 + g_2$ continuous

$\Rightarrow p \mapsto g_1(p) + g_2(p) = g(p) = ||f||_p$ is continuous

continuous $\Rightarrow p \mapsto ||f||_p$ is continuous. //
(3) We will use a change of variables to solve this problem. \[ \int_{(0,1)} \frac{1}{2} \int_{(0,1)} \frac{1}{y} f(y)^2 \, dy = \frac{1}{2} \| \frac{f(y)^2}{x} \|_1 \]

\[ \leq \frac{1}{2} \| f(y) \| \cdot \| f(y)^2 \|_B < +\infty \text{ if } \]

\[ q_B = 4, \quad \alpha < 2 \Rightarrow B \in (2,4] \]

\[ \Rightarrow q \in [2,4) \Rightarrow \text{These work.} \]

Consider \( f(y) = y^{-\alpha} \), then

\[ \| f(y)^2 \|_q = +\infty \text{ for } q \geq 2 \Rightarrow q \in [2,4). \]
(5) Suppose \( \forall x \in [0, 1] \), \( E \) is an interval \( I_x \) centered at \( x \) such that \( m(E \cap I) < \frac{m(I_x)}{8} \).

\( [0, 1] \) is compact and \( \exists I_x, x \in [0, 1] \), form an open cover \( \Rightarrow \exists \) a finite subcover which cover \( [0, 1] \). From this finite subcover, we can find \( I_1, \ldots, I_N \) such that \( I_m \) intersects only \( I_{m-1} \cup I_{m+1} \) and there union is \( [0, 1] \). Thus we have that

\[
m(E) \leq \sum_{n=1}^{N} m(E \cap I_n) < \sum_{n=1}^{N} \frac{m(I_n)}{8} \leq \frac{2m([0, 1])}{8} = \frac{1}{4} \Rightarrow \]

This proves the claim. \( \blacksquare \)
Analysis QR Part II

1) \( g(z) \) entire \( \iff \frac{\partial}{\partial z} (g) = 0 \).

Let \( g(z) = f(h(z)) \) where \( h(z) = 2z + \bar{z} \).

Then we have that

\[
\frac{\partial}{\partial z} \frac{f \circ h}{h} = \left( \frac{\partial f \circ h}{\partial z} \right) \frac{\partial h}{\partial z} + \left( \frac{\partial f \circ h}{\partial \bar{z}} \right) \frac{\partial \bar{z}}{\partial z}.
\]

But \( f \) entire \( \Rightarrow \frac{\partial f}{\partial z} = 0 \)

\[
\Rightarrow \left( \frac{\partial f \circ h}{\partial z} \right) \frac{\partial h}{\partial z} = 0. \quad \text{But,}
\]

\[
\frac{\partial h}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} (3x - iy) + i \frac{\partial}{\partial y} (3x - iy) \right)
\]

\[
= \frac{1}{2} [3 + 1] = 2
\]

\[
\Rightarrow \frac{\partial f \circ h}{\partial z} = 0. \quad \text{But,} \quad h(z) = 3x - iy
\]

\[
\Rightarrow \frac{\partial f}{\partial z} \circ h = 0 \iff \text{his bijective} \iff \frac{\partial f}{\partial z} \circ h = 0 \iff \frac{\partial f}{\partial z} = 0 \iff f \text{ constant}.
\]
Then, for $R$ big enough we have that all the zeros are contained in the contours since

$$p(iR) = R^8 - 10R^3 - 50R^2 + 1 \neq 0 \implies \text{No zeros on contour.} \quad p(z) \text{ has no poles so we have that } \int_C \frac{p'(z)}{p(z)} \, dz = 2\pi i \text{(N), where N is # of zeros counting multiplicity inside of C. as } R \to +\infty$$

$p(z)$ behaves like $z^8$ \implies between $-\pi/2 + i\pi/2$, $p(C)$ goes around 0 almost 4 times.

Since $p(t+i)$ are Re part $> 0 \quad \forall t \implies$
that $p(C)$ gets stuck and can't wrap around the origin anymore.

So, \[
\frac{1}{2\pi i} \int_{C} \frac{p'(z)}{p(z)} \, dz = \frac{1}{2\pi i} \int_{C} \frac{1}{z} \, dz
\]

\[= N = 4. \]
(3) Let \( f(z) = \ldots + a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + \ldots \)

Then \( f(z) + 2f(z^2) = \)

\[
(\ldots + a_{-3}z^{-3} + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + \ldots)
+ (\ldots + 2a_{-3}z^{-6} + 2a_{-2}z^{-4} + 2a_{-1}z^{-2} + 2a_0 + \ldots)
\]

\[\Rightarrow a_{-1} = a_{-3} = a_{-5} = \ldots = 0 \quad (\text{i.e. } a_{\text{odd}} = 0)\]

\[\Rightarrow a_{-2} + 2a_{-1} + a_{-2} = a_{-2} = 0\]

\[\Rightarrow 2a_{-2} + a_{-4} = a_{-4} = 0 \quad \text{except}\]

\[a_{-\text{even}} = 0 \Rightarrow f(z) \text{ does not have a singularity at } 0 \Rightarrow \text{No such } f \text{ exist.}\]
Claim: Every compact subset of $D$ is contained in $\alpha D$, $\alpha \ll 1$.

Proof. $\left\{ (1 - \frac{1}{n})D \right\}_{n=1}^{\infty}$ form an open cover of $X$ compact $\subset D$ (since it is a cover of $D$) $\Rightarrow$

3. a finite subcover $\Rightarrow X \subset (1 - \frac{1}{n})D$ some $n$ $\Rightarrow$ Claim. Let $X \subset \alpha D$ be compact.

Then $f \subset \alpha D$ is compact in $C$ $\Rightarrow$ Bounded $\Rightarrow$ $\left\{ f : \alpha D \rightarrow C \right\}_{n=1}^{\infty}$ is a set of holomorphic functions such that $f \subset \alpha D$ misses $\alpha$, $\beta$ for $n \geq M$, some $M$ since $Re f \rightarrow 0$ uniformly on $\alpha D$ $\Rightarrow$ $\left\{ f : \alpha D \rightarrow C \right\}_{n=M}^{\infty}$ form a normal family $\Rightarrow$ converge on $X$ to holomorphic function $f$, $f(z) = 0$, $Re f = 0$
Since $f$ is holomorphic $\Rightarrow \text{Im } f = 0$

$\text{Im } f_n \xrightarrow{\text{uniformly on compact sets}} 0$ $\Rightarrow$

$\text{Im } f_n \xrightarrow{\text{uniformly on compact sets}} 0$
Consider the keyhole contour.

Then,

\[ \int_{C} \frac{Z^+}{(Z+1)(Z+2)} \,dz = \]

\[ \int_{C_1} \frac{Z^+}{(Z+1)(Z+2)} \,dz + \int_{C_2} \frac{Z^+}{(Z+1)(Z+2)} \,dz + \int_{C_3} \frac{Z^+}{(Z+1)(Z+2)} \,dz + \int_{C_4} \frac{Z^+}{(Z+1)(Z+2)} \,dz = \]

\[ -2\pi i \left( (-1)^+ + \frac{Z}{(Z+1)(Z+2)} \right) \]

\[ = -1 \]
\[
\Rightarrow (1 - e^{2\pi i t}) \int_{\pi i t}^{\pi i t} \frac{z}{(z+1)(z+2)} \, dz
\]

\[
= 2\pi i \left[ 1 - z^+ \right] e^{-\pi i t + \pi i t}
\]

\[
\Rightarrow e^{-\pi i t} - e^{\pi i t} \int_{\pi i t}^{\pi i t} \frac{z^+}{(z+1)(z+2)} \, dz = [1 - z^+]
\]

\[
= -\sin(\pi t) \int_{\pi t+1}^{\pi t+2} \frac{x^+}{(x+1)(x+2)} \, dx = [1 - z^+]
\]

\[
\Rightarrow \frac{\pi \left[ 2^+ - 1 \right]}{\sin(\pi t)} = \int_{\pi t+1}^{\pi t+2} \frac{x^+}{(x+1)(x+2)} \, dx
\]