1. Problem 4

(1) Show that if $\mathbb{R}P^2$ of $K$, the Klein bottle, covers a manifold $N$ then $N$ is itself homeomorphic to $\mathbb{R}P^2$ or $K$, respectively.

Proof. First we will complete the problem for the case of $\mathbb{R}P^2$. Note that we have $\chi(\mathbb{R}P^2) = k \cdot \chi(N)$. But $\mathbb{R}P^2$ has a CW-complex structure consisting of one 0-cell, one 1-cell, and one 2-cell. It follows that $\chi(\mathbb{R}P^2) = 1 - 1 + 1 = 1$. Therefore, $k$ must be 1, and so we have that $\mathbb{R}P^2$ covers itself. Now we consider the case of the Klein Bottle $K$. $K$ has the structure of a CW-complex given by one 0-cell, two 1-cells, and one 2-cell. Therefore, $\chi(K) = 1 - 2 + 1 = 0$. It follows that $\chi(N) = 0$. But $N$ is a compact 2-manifold, which means that $N$ is either a connected sum of tori or a connected sum of $\mathbb{R}P^2$. If $N$ is a connected sum of $n$ tori, then $\chi(N) = 2(1 - n)$. If $N$ is the connected sum of $n$ $\mathbb{R}P^2$'s, then $\chi(N) = n - 2n + 2 = 2 - n$. Therefore, $N = \mathbb{R}P^2 \# \mathbb{R}P^2 = K$.

Is it possible for either $\mathbb{R}P^2$ or $K$ to nontrivially cover itself? Explain.

It is not possible for $\mathbb{R}P^2$ to cover itself since in the above we saw that $k = 1$. □

2. Problem 5

Let $X$ be a 1-point compactification of $S^3 \times \mathbb{R}$. Prove that $X$ is not a manifold and compute its homology groups $H_\ast(X, \mathbb{Z})$.

Notice that a one point compactification of $S^3 \times \mathbb{R}$ is given by $X = (S^3 \times [0, 1])/(S^3 \times \{0\} \cup S^3 \times \{1\})$. Let $p = S^3 \times \{0\}$ and consider an open neighborhood around $p$. An open neighborhood around $p$ will be homeomorphic to $\mathbb{D}^4 \land \mathbb{D}^4$ which is not homeomorphic to $\mathbb{R}^4$. Therefore, $X$ is not a manifold.

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3. Problem 6

Let $X$ be a compact Hausdorff space and let $X_1 \supset X_2 \supset \ldots$ be a sequence of closed connected subspaces. Prove that $\cap_{i=1}^{\infty} X_i$ is connected.

4. Problem 7

Let $V_1 \subset \mathbb{R}^4, V_2 \subset \mathbb{R}^4$ be linear subspaces, where the dimension of $V_j$ is $j, j = 1, 2$. Assume that $V_1 \not\subset V_2$, and let $X$

5. Problem 8

Let $M$ and $N$ be smooth connected manifolds, $M$ compact and $N$ non-compact. Prove that there is no submersion $f : M \rightarrow N$.

Since $M$ is compact $f(M)$ is compact by continuity. A compact subset of a Hausdorff space is closed. Also, submersions are open mappings. Therefore, $f(M)$ is an open and closed non-empty set of a connected manifold $N$. It follows that $f(M) = N$. But then $N$ is compact by continuity, and we thus have a contradiction.

6. Problem 10

Construct a connected CW-complex $X$ with $H_0(X, \mathbb{Z}) = \mathbb{Z}, H_1(X, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/10\mathbb{Z}$ and $H_2(X, \mathbb{Z}) = \mathbb{Z}^2$.

Figure 1. A CW-complex that suffices