1. Problem 1

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous map such that $f^{-1}([-n,n])$ is compact for each positive integer $n$. Show that $f$ achieves either a minimum value or a maximum value.

2. Problem 2

Construct a space $X$ with $H_0(X) = \mathbb{Z}$, $H_1(X) = \mathbb{Z}_2 \times \mathbb{Z}_3$, $H_2(X) = \mathbb{Z}$ and all other homology groups of $X$ vanishing.

Figure 1. From this drawing of the $CW$-complex structure all is clear.

3. Problem 6

Let $Z$ be the complement of the $x$-axis and the $y$-axis in $\mathbb{R}^3$, i.e.

$$Z = \mathbb{R}^3 - (x-axis \cup y-axis).$$

Compute the homology of $Z$.

Proof. It is clear that $Z$ deformation retracts to $S^2 \setminus \{(\pm 1, 0, 0), (0, 0, \pm 1)\}$. This then deformation retracts to $\mathbb{R}^2$ with 3 points removed, which further retracts to $S^1 \wedge S^1 \wedge S^1$. Therefore $H_n(Z) = 0$ for $n > 2$, $H_1(Z) = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, and $H_0(Z) = \mathbb{Z}$. 

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4. Problem 7

Let $X = \mathbb{R}P^2 - \text{point}$. Compute the homology of $X$.

Proof. It is clear that this deformation retracts to a circle, and therefore we have that $H_n(X) = 0$ for $n > 2$, and $H_1(X) = H_0(X) = \mathbb{Z}$. □

5. Problem 10

Let $X$ be a compact space and $f : X \to Y$ a continuous map to a Hausdorff space $Y$. Show that the induced map $f : X \to f(X)$ is a quotient map.

Proof. Since a closed subset of a compact space is compact, it follows that $f$ maps closed sets to compact sets. But a compact subset of a Hausdorff space is closed, so $f$ is a closed map and is therefore a quotient map. □