1. **Problem 1**

A space $X$ is constructed from two disjoint copies of $\mathbb{R}P^3$ and a copy of the unit interval $I$ by gluing one end of $I$ to a point of one copy of $\mathbb{R}P^3$, and gluing the other end of $I$ to the other copy of $\mathbb{R}P^3$.

1. Describe the universal cover $\tilde{X}$ of $X$.

The universal cover of $X$ is given by $\coprod_{n \in \mathbb{Z}} S^3_n \cup I_n/\sim$ where $\sim$ is the equivalence relation given by taking some $x_n \in S^3_n$ and $x_{n+1} \in S^3_{n+1}$ and identifying $0 \in I_n$ with $x_n$ and $1 \in I_n$ with $x_{n+1}$. Visually this creates a sorta beadlike looking pattern.

2. Compute the homology groups of $\tilde{X}$.

Since $\tilde{X}$ is the universal covering of $X$, it follows that $\tilde{X}$ is simply connected. Therefore we have that $H_0(\tilde{X}) = \mathbb{Z}$ and $H_n(\tilde{X}) = 0$ for $n > 0$.

2. **Problem 3**

Let $X$ denote the space $S^2 \cup A$, where $A = \{(x,0,0) \in \mathbb{R}^3 : 1 \leq x \leq 2\}$. Show that if $p : X \to Y$ is a covering map, then $p$ must be a homeomorphism, i.e. $X$ cannot cover anything except itself.

Consider an open neighborhood $U$ around $p((1,0,0))$. Then $p^{-1}(U)$ will contain a neighborhood around $(1,0,0)$ It is clear that no other open neighborhood of any point in $S^2 \cup A$ is homeomorphic to this neighborhood, and thus $p$ must be a 1-sheeted covering, and as such it cannot cover anything but itself.

3. **Problem 5**

Let $X$ denote the quotient space $\mathbb{R}/\mathbb{Q}$ of the real line obtained by identifying all the rationals to a single point. (This is not the group theoretic quotient.)

1. Is $X$ Hausdorff?

No, $X$ is not Hausdorff. For $x \in \mathbb{R}$, let $[x]$ be its equivalence class in $X$. I will show that the only open neighborhood around $[0]$ is $X$. 

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Proof. Let $U$ be an open set in $X$ around $[0]$. Then if $q : \mathbb{R} \to X$ is the quotient map in question, we have that $q^{-1}(U)$ is open in $\mathbb{R}$. In particular, $q^{-1}(U)$ contains $\mathbb{Q}$ which is a dense subset of $\mathbb{R}$. The only open subset of $\mathbb{R}$ containing $\mathbb{Q}$ is $\mathbb{R}$. Therefore, $U = X$. This proves that $X$ is not Hausdorff. \qed

(2) Is $X$ compact? Yes it is. We pretty much already proved this. The only open neighborhood around $[0]$ is $X$. Since any open cover contains an open set around $[0]$, we have proven the claim.

4. Problem 6

Identify the space of all $2 \times 2$ real matrices with $\mathbb{R}^4$ so that the matrix \[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\] corresponds to $(a, b, c, d)$. Show that the subspace $\sum$ of all matrices with determinant 1 is a smooth 3-dimensional manifold. Let $\Pi$ denote the hyperplane in $\mathbb{R}^4$ with the equation $x_1 + x_2 + x_3 - x_4 = 0$. Does $\Pi$ intersect $\sum$ transversely at $I$.

Consider the map $\det : \mathbb{R}^4 \to \mathbb{R}$ given by $\det(x_1, x_2, x_3, x_4) = x_1x_4 - x_2x_3$. The Jacobian of this map evaluated at an arbitrary point $(a_1, a_2, a_3, a_4)$ is given by $(a_4, -a_3, -a_2, a_1)$. This equals $(0, 0, 0, 0)$ iff $a_1 = a_2 = a_3 = a_4 = 0$. Therefore, 1 is a regular value of $\det$ from which we conclude that $\det^{-1}(1)$ is a smooth 3-dimensional submanifold of $\mathbb{R}^4$.

5. Problem 7

The suspension of a space $Y$ is the quotient space of $Y \times [0, 1]$ obtained by identifying $Y \times \{0\}$ to a point and separately identifying $Y \times \{1\}$ to a point. Let $X$ denote the suspension of $\mathbb{R}P^2$.

(1) Compute $\pi_1(X)$.

Let $U$ be the open set $\mathbb{R}P^2 \times [0, \frac{3}{4}]/\sim$ and let $V$ be the open set $\mathbb{R}P^2 \times [\frac{1}{4}, 1]/\sim$. It is clear that $U$ and $V$ both deformation retract to a point. Furthermore, it is clear that their intersection deformation retracts to $\mathbb{R}P^2$. It follows from the Van-Kampen theorem that $\pi_1(X) = 0$. 
