SOLUTIONS TO THE JANUARY 2, 2012 UNIVERSITY OF MICHIGAN QUALIFYING EXAM IN TOPOLOGY

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1. Problem 1

Let $X$ be a topological space, and $U$ an open subset of $X$. Let $A$ denote $U - \overline{U}$. Show that the interior of $A$ is empty.

Proof. First we will show that $A$ is closed. Let $x$ be a limit point of $A$. Then since $A \subset U$, it follows that $x \in U$. Furthermore, $x \notin U$ since $U$ is open. That is, there exists an open set $U_x \in U$ around $x$, and $A \cap U_x = \emptyset$. Therefore, $x \in A$, which implies that $A$ is closed, from which it follows that $A = \overline{A}$. Since the interior of $U$ is $U$, it follows that $\text{int}(A) = \emptyset$. □

2. Problem 2

(1) Show that $H_1(S^1 \times P^2) \cong \mathbb{Z} \times \mathbb{Z}_2$.

Note that $\pi_1(S^1 \times P^2) \cong \pi_1(S^1) \times \pi_1(P^2) \cong \mathbb{Z} \times \mathbb{Z}_2$. Since $H_1(X)$ is the abelianization of $\pi_1(X)$, we have that $H_1(X) \cong \mathbb{Z} \times \mathbb{Z}_2$.

(2) Suppose that $S^1 \times P^2$ covers some space, and let $h$ be a covering translation. Show that the induced isomorphism $h_*$ of $H_1(S^1 \times P^2)$ must be the identity.

Proof. Let $h$ be a covering translation of a covering $p$. Then, $h_* \circ p_* = p_*$. But $p_* = p_{1*} \times p_{2*}$ where $p_1$ is a homomorphism of $\mathbb{Z}$, and $p_{2*}$ is a homomorphism of $\mathbb{Z}_2$. Since neither of these are invertible, it follows that $h_*$ must be the identity. □

3. Problem 4

Let $f : X \to Y$ be a continuous map. Suppose that $Y$ is connected, and that $f^{-1}(y)$ is also connected, for each $y$ in $Y$.

(1) Show that if $f$ is a quotient map, then $X$ is connected. Suppose that $X$ is not connected. Suppose that $X = U \cup V$ where $U$ and $V$ are disjoint open sets. Since $F_y = f^{-1}(y)$ is connected for each $y \in Y$, it follows that $F_y \subset U$ or $F_y \subset V$. Now, let $U' = \{y \in Y \mid F_y \subset U\}$ and let $V' = \{y \in Y \mid F_y \subset V\}$. Since $f$ is a quotient map, it follows that $U'$ is open since $f^{-1}(U') = U$, and likewise $V'$ is open since $f^{-1}(V') = V$. Since $Y = U' \cup V'$, and $U'$ and $V'$ are disjoint open sets, it follows that $Y$ is disconnected which is a contradiction.

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4. Problem 5

A space \( X = \bigcup_{k=1}^{\infty} X_k \), where each \( X_k \) is a simply connected subset of \( X \), and \( X_k \subset X_{k+1} \), for each \( k \geq 1 \).

(1) Show that if each \( X_k \) is open in \( X \), then \( X \) is simply connected.

Let \( p : S^1 \rightarrow X \) be a path. We wish to show \( p \) is null-homotopic. Since \( S^1 \) is a compact space, by continuity, we have that \( p(S^1) \) is compact in \( X \). This implies that \( p(S^1) \) is completely contained in one of the \( X_k \). Since each of the \( X_k \) is simply connected, it follows that \( X \) must be simply connected.

(2) Show that if each \( X_k \) is closed in \( X \) (but not necessarily open), then \( X \) need not be simply connected.

Let \( X_k \) be \( \frac{k}{k+1} \) of a unit circle starting at \((0, 1)\) and going counterclockwise. Then the \( X_k \) satisfy the hypothesis, and \( X = S^1 \) which is not simply connected.

5. Problem 6

A space \( X \) is constructed by gluing a Moebius band \( M \) and an annulus \( A \) as follows. One boundary component of \( A \) is glued to \( \partial M \) by a homeomorphism. The other boundary component of \( A \) is glued to the core circle of \( M \) by a homeomorphism.

Calculate the homology groups of \( X \).

![Figure 1. CW-complex structure](image)

**Proof.** \( \partial_2(e_1^2) = 2e_1^1 + e_2^1 - e_3^1 \), \( \partial_2(e_2^2) = 0 \) and \( \partial_1 = 0 \). Therefore, \( H_2(X) = \mathbb{Z} \) and \( H_1(X) = \langle e_1^1, e_2^1, e_3^1 \rangle / \langle 2e_1^1 + e_2^1 - e_3^1 \rangle = \mathbb{Z} \times \mathbb{Z} \). Since \( X \) is connected, we have that \( H_0(X) = \mathbb{Z} \). \( \square \)

6. Problem 7

Let \( X \) denote the cone on the real line \( \mathbb{R} \). Decide whether \( X \) is locally compact. [The cone on a space \( Y \) is the quotient of \( Y \times I \) obtained by identifying \( Y \times \{0\} \) to a point.]

\( X \) is locally
7. Problem 8

Let $M$ be a connected smooth compact $n$-manifold without boundary, and let $N$ be a connected smooth $n$-manifold. Let $f : M \to N$ be a smooth immersion.

(1) Show that $f$ is a covering map.

First we will show that $f$ is surjective. Since $M$ is compact, $f(M)$ is compact by continuity. Since $N$ is Hausdorff, $f(M)$ is closed. Also, immersions are local diffeomorphisms, which in turn are local homeomorphisms, and consequently open maps. Therefore $F(M)$ is open in $N$. Since $N$ is connected, and $M$ is nonempty, it follows that $f(M) = N$, and therefore $f$ is surjective. Since $f$ satisfies the Hypothesis of Jan 2013 test, we are finished.

(2) Show that $S^2 \times S^2$ cannot be immersed into $\mathbb{R}^4$, but can be embedded in $\mathbb{R}^5$.

By the first part, if $f$ is a smooth immersion of $S^2 \times S^2$ into $\mathbb{R}^4$, then $f$ is a covering map as well, which implies that $f(S^2 \times S^2) = \mathbb{R}^4$. But, this would mean $S^2 \times S^2$ is compact which cannot be true.

8. Problem 10

If $A$ is a subspace of a topological space $X$, a map $f : X \to A$ is a retraction if the restriction of $f$ to $A$ is the identity map. Prove that

(1) if $X$ is a compact smooth manifold, there is no retraction of $X$ to its boundary.

Proof. Suppose that there exists a retraction $r : M \to \partial M$. By the regular value theorem, there is $p \in \partial M$ such that $r^{-1}(p)$ is a smooth 1-dimensional manifold of $M$. Since $r$ is continuous, and $p$ closed, we have that $r^{-1}(p)$ is a closed subset of compact space and hence is compact itself. Clearly $\partial r^{-1}(p) = \{p\}$ and so we have that $|\partial r^{-1}(p)| = 1$. However, by classification of compact 1-manifolds, we have that up to diffeomorphism $r^{-1}(p)$ is a disjoint union of unit intervals and copies of $S^1$. This implies that $|\partial r^{-1}(p)|$ is even and hence we have a contradiction. □

(2) there is no retraction of $\mathbb{R}P^2$ onto $\mathbb{R}P^1$.

Proof. Suppose that there was such a retraction. This would mean that there existed a continuous deformation from $\mathbb{R}P^2$ to $\mathbb{R}P^1$. However, the fundamental group of a space $x$ is not affected by a continuous deformation. Since $\pi_1(\mathbb{R}P^2) = \mathbb{Z}_2$, and $\pi_1(\mathbb{R}P^1) = 0$, it follows that there does not exist a retraction between the two spaces. □

(3) there is no retraction of the plane $\mathbb{R}^2$ onto the “topologist’s sine curve”, where $W = \{(x, \sin \frac{1}{x}) : x \neq 0\} \cup \{(0, y) : -1 \leq y \leq 1\}$. 
Proof. Note that the continuous image of a path connected space is path connected. Therefore, the image of $\mathbb{R}^2$ cannot be $W$ since $W$ is not path connected. This proves the claim.

$\Box$