1. \[
\lim_{x \to \infty} \int_{x}^{x+1} f'(x) \, dx =
\]

\[
\lim_{x \to \infty} [f(x+1) - f(x)] = a - a = 0.
\]

But, by the mean value theorem we have that \[
\int_{x}^{x+1} f'(x) \, dx = f'(\xi) \text{ for some } \xi \in [x, x+1]
\]

\[
\Rightarrow \lim_{x \to \infty} f'(\xi) = 0
\]

\[
= \lim_{x \to \infty} f'(x) = b \Rightarrow b = 0.44
\]
2. Note that
\[
\lim_{R \to \infty} \int_0^R \frac{\sin x}{x} \, dx = \lim_{R \to \infty} \int_0^R \int_0^{R \cdot \infty} e^{-xy} \sin x \, dy \, dx
\]

Now since \( \int_0^\infty \int_0^\infty e^{-xy} \sin x \, dy \, dx \)
\[
= \int_0^\infty \left| \frac{\sin x}{x} \right| \, dx < \infty \quad \text{since}
\]

\( |\sin x|/x \) is continuous on \( \mathbb{R} \) and we have that the continuous image of \([0, R]\) is compact \( \Rightarrow |\sin x|/x \) attains a maximum on \([0, R]\), say \( M \Rightarrow \int_0^\infty \left| \frac{\sin x}{x} \right| \, dx \leq RM \)

\( \Rightarrow \) We can invoke Fubini on \([0, R] \times (0, \infty)\)
Thus, \( \lim_{R \to \infty} \int_0^\infty \int_0^R e^{-xy} \sin x \, dy \, dx = \)
\[ \lim_{R \to \infty} \int_0^\infty \int_0^R e^{-xy} \sin x \, dy \, dx \]
\[= \lim_{R \to \infty} \int_0^\infty \left( \frac{-1}{1 + y^2} (y e^{-xy} \sin x + e^{-xy} \cos x) \right) dy \]
\[= \lim_{R \to \infty} \int_0^\infty \left( \frac{-1}{1 + y^2} - \frac{e^{-Ry}}{1 + y^2} (y \sin R + \cos R) \right) dy \]

We wish to show \( \lim_{R \to \infty} \int_0^\infty \left( \frac{e^{-Ry}}{1 + y^2} (y \sin R + \cos R) \right) dy = 0 \)

Let \( f_R(y) = \frac{e^{-Ry}}{1 + y^2} (y \sin R + \cos R) \).

Then we have \( f_R(y) \leq \frac{e^{-y}}{1 + y^2} (y + 1) \)

for \( R > 1 \)
But, \( e^{-y} \leq e^{-y} \leq \frac{e^{-y}}{1+y^2} \) 

\[ \frac{e^{-y}}{1+y^2} \leq e^{-y} \Rightarrow (*) \leq 2e^{-y} \]

But, \[ \|2e^{-y}\|_{L^1(0,\infty)} = -2e^{-y}\big|_0^\infty = 2 \]

\[ \Rightarrow 2e^{-y} \text{ dominates the } f_k, \text{ so we can apply the Dominated Convergence Theorem and so,} \]

\[ \lim_{R \to \infty} \int_0^\infty e^{-Ry} \left( y \sin R + \cos R \right) dy \]

\[ = \int_0^\infty \lim_{R \to \infty} e^{-Ry} \left( y \sin R + \cos R \right) dy \]

\[ = \int_0^\infty dy = 0 \]
It now follows that the integral we originally wanted to solve

\[ \int_0^\infty \frac{1}{1+y^2} \, dy = \tan^{-1}(y) \bigg|_0^\infty \]

\[ = \frac{\pi}{2} . \]
Let \( E = \{ x \in X : |f(x)| > t \} \)

3. We have that \( (\Delta) \)

\[
\int_0^\infty \int_{E} |f(x)| \, dx \, dt = \int_{(0, \infty) \times X} 1_E(x) dtdx
\]

Now, \( 1_E(x) \) is a positive nonnegative measurable on \((0, \infty) \times X\) ⇒ We can invoke Fubini.

Thus, we have \( (* \Delta) = \)

\[
\int_0^\infty \int_X 1_E(x) \, dx \, dt + \int_X |f(x)| \, dx \, dt
\]

\[= \int_X |f(x)| \, dx \]
Going by to (A), we have

\[ \int f(x) \, dx = \sum_{x \in X : |f(x)| > t} dt + \sum_{x \in X} |f(x)| dt \]

\[ \Rightarrow 1 \mu(x) + \int \frac{K}{t^p} \, dt \]

\[ = \mu(x) + \frac{t^{1-p} K}{1-p} \bigg|_{t}^{\infty} \]

\[ = \frac{K}{p-1} + \mu(x) < +\infty \]

\[ \Rightarrow \| f \|_1, z + \infty \Rightarrow f \in L_1. \]
4. By the Lebesgue Differentiation Theorem, we have that

\[ \lim_{|I_x| \to 0} \frac{1}{|I_x|} \int_{I_x} |1_{E}(y) - 1_{E}(x)| \, dy = 0 \]

for a.e. \( x \in [0,1] \) since \( 1_{E} \) is Lebesgue Integrable.

Note that (*) \( \geq \)

\[ \lim_{|I_x| \to 0} \frac{1}{|I_x|} \left| \int_{I_x} 1_{E}(y) \, dy - \int_{I_x} 1_{E}(x) \, dy \right| \]

If \( x \notin E \), then the limit reduces to

\[ \lim_{|I_x| \to 0} \frac{1}{|I_x|} \int_{I_x} 1_{E}(y) \, dy = 0 \]

\[ \lim_{|I_x| \to 0} \frac{m(\overline{E \cap I_x})}{|I_x|} = 0 \text{ for a.e. } x \notin E \]

But, by assumption, \( m(\overline{E \cap I_x}) \geq \alpha m(I_x) \)

\[ \forall I_x \Rightarrow \lim_{|I_x| \to 0} \frac{m(\overline{E \cap I_x})}{|I_x|} \geq \alpha \neq 0 \]

\[ \Rightarrow x \notin E \text{ for at most } \ldots \text{Lebesgue points} \Rightarrow m(E) = 1 \]
5. Let $X = [0, 1]$ and let $\mu$ be a normal measure. Define $f_n = n^2 1_{[0, \frac{1}{n}]}$

then, $\lim_{n \to \infty} f_n = 0$ which is integrable as are the $f_n$. So,

$\int f_n \, d\mu = \| f_n \|_1 = n \Rightarrow \lim_{n \to \infty} \| f_n \|_1 = \infty$

This proves the claim!

(b) We have $\lim_{n} \| f_n - f \|_1 \geq \lim_{n} \int (f_n - f) \, d\mu$

$= \lim_{n} \left| \int f_n \, d\mu - \int f \, d\mu \right| = 0$ by assumption

$\Rightarrow \lim_{n} \| f_n - f \|_1 = 0$, i.e. $f_n \to f$ in $L_1$. \(/\)
(1) \( f: \mathbb{H} \rightarrow \Delta \), \( f(5) = 0 \) can be written as \( f = g \circ h \).

Where \( h: \mathbb{H} \rightarrow \Delta \) is a conformal map sending \( 5 \rightarrow 0 \) and \( g: \Delta \rightarrow \Delta \) is an arbitrary analytic function with \( g(0) = 0 \).

Then, \( f'(5) = g'(0) h'(5) \).

By Schwartz, \( |g'(0)| \leq 1 \).

Taking \( g(z) = z \), we obtain the max.

So, \( |f'(5)| = |h'(5)| \)
\[ H \rightarrow B \]

\[ e^{\frac{\pi}{2} i} \]

\[ \frac{z - B}{i (z - B)} \]

\[ \frac{z - 5i}{i z + 5i} \]
The map $z \mapsto iz$ rotates $\mathbb{H}_R \mapsto \mathbb{H}_u$
(i.e. the upper half plane) also it's clearly conformal.

Now, $\frac{z-B}{z-B}$ mapping $\mathbb{H}_u \mapsto \mathbb{D}$ conformal for $B \in \mathbb{H}_u$, then $\frac{iz-B}{iz-B}$ mapping $\mathbb{H}_R \mapsto \mathbb{D}$

Choose $B=iS$, then $B \in \mathbb{H}_u$ and

$\frac{iz-ib}{iz+ib} = \frac{z-S}{z+S}$ is a map taking $\mathbb{H}_R \mapsto \mathbb{D}$
$\frac{z+S}{z-S} = a(z)$

with $a(S) = 0$. $|a'(S)| = \frac{10}{18} = \frac{1}{10}$

So by Riemann Mapping Thm we have that

$|f'(S)| = \frac{1}{10}$. /
(2) Take $u^2$, then $(u^2)_{xx} + (u^2)_{yy} = 0$

on $R$. But, $(u^2)_{xx} + (u^2)_{yy} = (2u u_x)_x + (2u u_y)_y = 2u u_{xx} + 2u_x^2 + 2u u_{yy} + 2u_y^2 = 0$ on $R$.

$2u_x^2 + 2u_y^2 \Rightarrow u_x = u_y = 0$ on $R$.

Since $f = u + iv$ is entire $\Rightarrow$

$f'(z) = u_x + iv_x$. But, we have $v_x = -u_y = 0$ on $R$ $\Rightarrow f'(z) = 0$ on $R$.

So, by the Identity principle, since $f'(z) = 0$ on a sequence with limit point or since it's $0$ on $R$, either way we $f' = 0$ on $C$. $\Rightarrow f$ is constant. $\Box$
(3) Since $1 - \cos z$ and $100z^2$ are both holomorphic on $C$ and since $100z^2$ is nonvanishing on the unit circle $S^1$, we can invoke the identity principle. Since $\max_{S^1} |100z^2| = 100 > \max_{S^1} |1 - \cos z| \leq 20$, $100z^2$ and $100z^2 + (1 - \cos z)$ have the same number of roots on $1D$, namely 2. //
Continued. Now for finding the 2 solutions.

$\cos z = 1 + 100z^2 \Rightarrow$

$1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = 1 + 100z^2$

$\Rightarrow -\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = 100z^2$

$\Rightarrow (200 + 1)z^2 - \frac{z^4}{2!} + \frac{z^6}{4!} - \frac{z^8}{6!} + \cdots = 0$

$\Rightarrow 0 \text{ is a solution with multiplicity 1, and so we are done.}$
(4) Showing \( I(t, y) \) is well-defined can be accomplished by showing the integral is finite since

\[
f(x+iy)e^{-ixt} = \left[ U(x+iy) + iV(x+iy) \right] e^{-ixt}
\]

where \( U \) and \( V \) are harmonic and thus are continuous and differentiable. IE

\[
\frac{d}{dx} U(x+iy) = U_x(x+iy) \text{ and } \frac{d}{dx} V(x+iy) = V_x(x+iy)
\]

\[\Rightarrow\text{ integral makes sense if it is finite}\]

\[
\left| \int_R f(x+iy) e^{\pm x} \, dx \right| \leq \int_R |f(x+iy)| \, dx
\]

\[
\leq C \int_R \frac{1}{1+x^2} \, dx = C \tan^{-1}(x) \Big|_{-\infty}^{\infty} = \pi C.
\]
a) I just showed its bounded.
(a) We show that \( I(t, y) \) is continuous.

\[
\lim_{h \to 0} |I(t+h, y) - I(t, y)|
\]

\[
\leq \lim_{h \to 0} \int_{\mathbb{R}} \left| \frac{e^{-ixh}}{1+x^2} - 1 \right| \, dx
\]

\[
\leq \lim_{h \to 0} 2C \int_{\mathbb{R}} \frac{1}{1+x^2} \, dx = 2\pi C
\]

Here we used \( 2 \leq |e^{-ixh} - 1| \)

So \( 2C \) dominates the \( f_h \).

+ By Dominated Convergence Thm

\[
\lim_{h \to 0} \int_{\mathbb{R}} \left| \frac{e^{-ixh}}{1+x^2} - 1 \right| \, dx = \int_{\mathbb{R}} 0 \, dx = 0
\]

\( \Rightarrow \) \( I(t, y) \) is continuous in first coordinate.
5) Suppose 0 is a removable singularity

\[ \lim_{z \to 0} \left| \frac{f(z)}{e^{f(z)}} \right| = \lim_{z \to 0} e^{f(z)} = e^{f(0)} \]

\[ \Rightarrow \text{z=0 is Not Pole. Now, suppose} \]

\[ f(z) \text{ has a pole of order n} \]

\[ f(z) = p_n(z^{-1}) + g(z) \]

\[ \Rightarrow e^{f(z)} = e^{p_n(z^{-1}) + g(z)} \]

Where \( g(z) \) is entire, \( p_n(z) \) is a polynomial
of degree n, we group the constant term
with \( p_n(z^{-1}) \)

\[ \lim_{z \to 0} g(z) = 0 \]
First, note that $e^{p_n(z)}$ is not a polynomial since $p_n(z)$ nonconstant entire $\Rightarrow p_n(z)$ takes on one of $\mathbb{Z} \cup i\mathbb{R}$ or $e$ is nonconstant entire $\Rightarrow$ Never is zero $\Rightarrow$ Not a nonconstant polynomial $\Rightarrow$ $e^{p_n(z)}$ has an essential singularity at $z = 0$.

$\Rightarrow e^{p_n(\frac{1}{z})}$ has an essential sing at 0 since $\lim_{z \to 0} e^{f(z)} = \lim_{z \to 0} e^{p_n(z^{-1})} g(z)$

$= \lim_{z \to 0} e^{p_n(z^{-1})} \Rightarrow z = 0$ is an essential singularity.
Now, suppose that \( z=0 \) is an essential singularity of \( f(z) \Rightarrow \) in any nbd \( \mathbb{D} \neq 0 \), say \( N_0 \), we have \( f(N_0) = \mathbb{C} \) minus possibly 1 pt.

So, we note that \( e^z \) is entire so

\[
\exp(f(N_0)) = \exp(\mathbb{C} \setminus \text{maybe one pt})
\]

\[
= \mathbb{C} \setminus \{0\} \quad \text{(maybe one other point)}
\]

(We know it is \( \mathbb{C} \setminus \{0\} \) since takes all but possibly one pt \( \infty \) many times.

So By Weierstrass Essential

\[
\Rightarrow \text{Can't be Pole.}
\]