Some results concerning Strongly Summable Ultrafilters on Abelian Groups

David J. Fernández Bretón

Department of Mathematics and Statistics
York University

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The Čech-Stone compactification of a discrete abelian group \((G, +)\) is the set \(\beta G\) of ultrafilters on \(G\), with basic open sets of the form

\[ \bar{A} = \{ p \in \beta G \mid A \in p \} \quad (A \subseteq G). \]

Every \(x \in S\) is identified with

\[ \{ A \subseteq G \mid x \in A \}, \]

and the group operation \(+\) on \(G\) is extended by the formula

\[ p + q = \{ A \subseteq G \mid \{ x \in G \mid A - x \in q \} \in p \}, \]

and \(G^* = \beta G \setminus G\) is a closed subsemigroup.
Denote by $\vec{x} = \langle x_n | n < \omega \rangle$ a sequence (typically injective) of elements of $\mathcal{G}$.

$$
\text{FS}(\vec{x}) = \left\{ \sum_{n \in a} x_n \bigg| a \in [\omega]^\omega \setminus \{\emptyset\} \right\}.
$$

**Definition**

We say that $p \in \mathcal{G}^*$ is **strongly summable** if for every $A \in p$ there exists a sequence $\vec{x}$ such that $p \ni \text{FS}(\vec{x}) \subseteq A$.

(i.e. $p$ has a basis of FS-sets)
Theorem (Hindman-Blass on $\mathbb{Z}$, Hindman-Protasov-Strauss in general)

*Every strongly summable ultrafilter $p$ is an idempotent (i.e. $p = p + p$).*

Theorem (Hindman-Strauss)

*Let $p \in \mathbb{Z}^*$ be a strongly summable ultrafilter, and let $q, r \in \omega^*$ be such that $q + r = r + q = p$. Then $q, r \in \mathbb{Z} + p$.*

Theorem (Hindman-Protasov-Strauss)

*If $G \subseteq \mathbb{T} = \mathbb{R}/\mathbb{Z}$, and $p \in G^*$ is strongly summable, then whenever $q, r \in G^*$ are such that $q + r = r + q = p$, it must be the case that $q, r \in G + p$.*
Properties of Strongly Summable Ultrafilters

Definition

We say that \( p \in G^* \) has the **trivial sums property** if whenever \( q, r \in G^* \) are such that \( q + r = p \), we must have that \( q, r \in G + p \).

Definition (Hindman-Protasov-Strauss)

An ultrafilter \( p \in G^* \) is **sparse** if for every \( A \in p \) there exist a sequence \( \vec{x} = \langle x_n \mid n < \omega \rangle \) and a moiety \( M \) of \( \omega \) such that \( \text{FS}(\vec{x}) \subseteq A \) and \( \text{FS}(\langle x_n \mid n \in M \rangle) \in p \).

Theorem (Hindman-Protasov-Strauss)

If \( G \subseteq \mathbb{T} \) and \( p \in G^* \) is sparse, then \( p \) has the trivial sums property.
Theorem (Hindman-Steprāns-Strauss)

We can assume that $G \subseteq \bigoplus_{n<\omega} T$. If $p \in G^*$ is a strongly summable ultrafilter, and

$$\left\{ x \in S \mid \pi_{\rho(x)}(x) \neq \frac{1}{2} \right\} \in p, \quad (\rho(x) = \min\{i < \omega \mid \pi_i(x) \neq 0\})$$

then $p$ is sparse.

Theorem (F.B.)

No matter what $G$ is, if $p \in G^*$ is strongly summable then it is sparse and it has the trivial sums property.
Definition (Blass)

A nonprincipal ultrafilter $p$ on $[\omega]^\omega$ is called a **union ultrafilter** if for every $A \in p$ there exists a pairwise disjoint sequence $\vec{x} = \langle x_n | n < \omega \rangle$ of elements of $[\omega]^\omega$ such that $p \ni \text{FU}(\vec{x}) \subseteq A$.

Definition (Hindman-Blass)

We will say that a strongly summable ultrafilter $p \in G^*$ is **additively isomorphic to a union ultrafilter** if for some sequence $\vec{x}$ of elements of $G$ with $\text{FS}(\vec{x}) \in p$, the mapping $\sum_{i \in a} x_i \mapsto a$ sends $p$ to a union ultrafilter.
Theorem (Blass-Hindman for \( \mathbb{Z} \), F.B. in general)

If \( p \in G^* \) and \( \{ x \in G \mid 2x = 0 \} \notin p \) then \( p \) is additively isomorphic to a union ultrafilter.

Theorem (F.B.)

Assume \( \text{cov}(\mathcal{M}) = \mathfrak{c} \). Then, there exists a strongly summable ultrafilter on the Boolean group \( ([\omega]^<\omega, \triangle) \) which is not additively isomorphic to a union ultrafilter.
Theorem (Hindman-Blass for \( \mathbb{Z} \), Hindman-Protasov-Strauss in general)

Given \( G \), there exists a mapping \( \mu : G \rightarrow \omega \) such that, if \( p \in G^* \) is strongly summable, then \( \mu(p) \) is a P-point. Furthermore, if \( \{ x \in G \mid 2x = 0 \} \notin p \), then \( \mu(p) \) is rapid.

Theorem (Krautzberger)

If \( p \in \mathbb{Z}^* \) then \( p \) is rapid. (Hence, if \( p \in G^* \) and \( \{ x \in G \mid 2x = 0 \} \notin p \), then \( p \) is rapid).

Theorem (F.B.)

Let \( \max : [\omega]^\omega \rightarrow \omega \). If \( p \) is a strongly summable ultrafilter on the Boolean group \( ([\omega]^\omega, \triangle) \), then both \( \max(p) \) and \( p \) itself are rapid.
If $p$ is a strongly summable ultrafilter on $G = ([\omega]^{<\omega}, \triangle)$, let

$$\text{Pr}(p) = \{(A, \text{FS}(X)) | A \in [G]^{<\omega} \land F\triangle(X) \in p\}$$

with $(A, \text{FS}(X)) \leq (B, \text{FS}(Y))$ iff $A \supseteq B$ and $\text{FS}(X \cup (A \setminus B)) \subseteq F\triangle(Y)$ (equivalently $X \cup (A \setminus B) \subseteq \text{FS}(Y)$).

**Theorem (F.B.)**

Let $\omega < \lambda < \kappa$ be two regular cardinals. A finite support iteration of length $\lambda$, with iterands of the form $\text{Pr}(p) \star \text{Fn}(\kappa, 2)$ ($\text{Fn}(\kappa, 2)$ is the forcing notion that adds $\kappa$ many Cohen reals) yields a model where $\text{cov}(\mathcal{M}) = \lambda$, $c = \kappa$ and there exists a strongly summable ultrafilter on $G$ (actually, this ultrafilter is generated by $\lambda$ elements, so $u = \lambda$ as well).


