Math 452 - Advanced Calculus II

Homework # 4. Due Mar. 20, 2015, noon

Q 1. For each positive integer \( n \), the Bessel function \( J_n(x) \) may be defined by:

\[
J_n(x) = \frac{x^n}{1 \cdot 3 \cdot 5 \cdots (2n-1)\pi} \int_{-1}^{1} (1 - t^2)^{n-1/2} \cos xt \, dt.
\]

Prove that \( J_n(x) \) satisfies Bessel’s differential equation

\[
J''_n + \frac{1}{x} J'_n + \left( 1 - \frac{n^2}{x^2} \right) J_n = 0.
\]

Q 2. Let \( A \subset \mathbb{R}^m \) and \( B \subset \mathbb{R}^n \) be contented sets, and \( f : \mathbb{R}^m \to \mathbb{R} \) and \( g : \mathbb{R}^n \to \mathbb{R} \) integrable functions. Define \( h : \mathbb{R}^{m+n} \to \mathbb{R} \) by \( h(x, y) = f(x)g(y) \), and show that

\[
\int_{A \times B} h = \left( \int_A f \right) \left( \int_B g \right).
\]

Conclude as a corollary that \( v(A \times B) = v(A)v(B) \).

Q 3. Consider the \( n \)-dimensional solid ellipsoid

\[
E = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^{n} \frac{x_i^2}{a_i^2} \leq 1 \right\}.
\]

Note that \( E \) is the image of the unit ball \( B_1 \) (measured in the euclidean norm) in \( \mathbb{R}^n \) under the map \( T : \mathbb{R}^n \to \mathbb{R}^n \) defined by:

\[
T(x_1, \ldots, x_n) = (a_1x_1, \ldots, a_nx_n).
\]

Show that \( v(E) = a_1a_2 \cdots a_nv(B_1) \).
Q 4. Let $R$ be the solid torus in $\mathbb{R}^3$ obtained by revolving the circle $(y-a)^2 + z^2 \leq b^2$, in the $yz$-plane, about the $z$-axis. Note that the mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$
\begin{align*}
    x &= (a + w \cos v) \cos u, \\
    y &= (a + w \cos v) \sin u, \\
    z &= w \sin v,
\end{align*}
$$

maps the interval $Q = \{(u, v, w) : u, v \in [0, 2\pi] \text{ and } w \in [0, b]\}$ onto $R$. Apply the change of variables formula to calculate the volume of this torus.

Q 5. Find the volume of $B_r$, the ball of radius $r$ (measured in the euclidean norm) in $\mathbb{R}^n$. HINT: See exercise 5.17 in chapter 4 of the textbook.