Q 1. (10 points) Given \( a \in \mathbb{R}^n \), denote by \( L_a \) the linear function

\[
L_a(x) = a \cdot x = \sum_{i=1}^{n} a_i x_i.
\]

Consider the norms of \( L_a \) with respect to the sup norm \( \| \cdot \|_\infty \) and the 1-norm \( \| \cdot \|_1 \) on \( \mathbb{R}^n \). Prove that \( \| L_a \|_1 = |a|_\infty \), while \( \| L_a \|_\infty = |a|_1 \).

SOLUTION: To show that \( \| L_a \|_1 = |a|_\infty \), notice that for any \( x \in \mathbb{R}^n \):

\[
|L_a(x)| = \left| \sum_{i=1}^{n} a_i x_i \right| \leq \sum_{i=1}^{n} |a_i x_i| \leq \max_{1 \leq i \leq n} \{ |a_i| \} \sum_{i=1}^{n} |x_i| = |a|_\infty |x|_1.
\]

Thus, in particular for \( x \in \mathbb{R}^n \setminus \{0\} \):

\[
\left| L_a \left( \frac{x}{|x|_1} \right) \right| \leq |a|_\infty \quad \text{and} \quad \max_{x \in \partial B_1^1} |L_a(x)| \leq |a|_\infty,
\]

that is, \( \| L_a \|_1 \leq |a|_\infty \). On the other hand, if \( |a|_\infty = |a_k| \) for some \( 1 \leq k \leq n \), let \( \tilde{x} = \varepsilon_k e_k \), where \( \varepsilon_k = a_k/|a_k| \) and \( e_k \) is the the \( k \)-th unit vector in the standard basis of \( \mathbb{R}^n \). Clearly \( \tilde{x} \in \partial B_1^1 \) and

\[
|a|_\infty = |a_k| = a_k \varepsilon_k = L_a(\tilde{x}) \leq \max_{x \in \partial B_1^1} |L_a(x)|.
\]

That is \( |a|_\infty \leq \| L_a \|_1 \). Hence \( \| L_a \|_1 = |a|_\infty \). To show that \( \| L_a \|_\infty = |a|_1 \), notice that \( L_a(x) = Ax \), where \( A \) is the \( 1 \times n \) matrix defined as

\[
A = [a_1 \ a_2 \ \cdots \ a_n].
\]

From one the results discussed in class, \( \| A \| = \| L_a \|_\infty = |a|_1 = \sum_{j=1}^{n} |a_j| = |a|_1 \).
Q 2. (10 points)

(a) Show that the linear mapping $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is one-to-one if and only if $a = \min_{x \in \partial B^\infty_1} |T(x)|_\infty$ is positive.

(b) Conclude that the linear mapping $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ is one-to-one if and only if there exists $a > 0$ such that $|T(x)|_\infty \geq a|x|_\infty$ for all $x \in \mathbb{R}^n$.

Solution:

(a) ($\Rightarrow$) Suppose that $T$ is one-to-one. For a contradiction, assume that $a = \min_{x \in \partial B^\infty_1} |T(x)|_\infty = 0$. Since the map $|T(\cdot)|_\infty$ is continuous and $\partial B^\infty_1$ is compact, $\exists \tilde{x} \in \partial B^\infty_1$ such that $|T(\tilde{x})|_\infty = 0$. Then $T(\tilde{x}) = 0 \Rightarrow \tilde{x} = 0$, since $T$ is one-to-one. However, $|\tilde{x}|_\infty = 1 \Rightarrow \tilde{x} \neq 0$. Hence, a contradiction. Therefore, $a > 0$.

($\Leftarrow$) Suppose now that $a > 0$. We want to show that $|T(x)|_\infty > 0$ whenever $x \neq 0$. If $a = \min_{x \in \partial B^\infty_1} |T(x)|_\infty > 0$, it means that for any $x \in \mathbb{R}^n \setminus \{0\}$,

$$ \left| T\left( \frac{x}{|x|_\infty} \right) \right|_\infty \geq a > 0 $$

Therefore $T(x) \neq 0 \Leftrightarrow x \neq 0$, or $T(x) = 0 \Rightarrow x = 0$. That is, $T$ is one-to-one.

(b) ($\Rightarrow$) From (a), if $T$ is one-to-one, then the quantity $a = \min_{x \in \partial B^\infty_1} |T(x)|_\infty$ is positive, and for any $x \in \mathbb{R}^n \setminus \{0\}$,

$$ \left| T\left( \frac{x}{|x|_\infty} \right) \right|_\infty \geq a \Rightarrow |T(x)|_\infty \geq a|x|_\infty. $$

Clearly, $|T(x)|_\infty = 0 = a|x|_\infty$ for $x = 0$, so the inequality holds for any $x \in \mathbb{R}^n$.

($\Leftarrow$) If $\exists a > 0$ such that $|T(x)|_\infty \geq a|x|_\infty$ for all $x \in \mathbb{R}^n$, then:

$$ 0 \leq |x|_\infty \leq \frac{1}{a}|T(x)|_\infty, $$

and $T(x) = 0 \Rightarrow x = 0$. That is, $T$ is one-to-one.
Q 3. (10 points) Suppose that the equation $f(x, y, z) = 0$ can be solved for each of the three variables $x, y, z$ as a differentiable function of the other two. Prove that
\[ \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} \frac{\partial z}{\partial x} = -1. \]
Verify this in the case of the ideal gas equation $pv = RT$, where $p, v, T$ are the variables and $R$ is a constant.

Solution: From the hypothesis of the problem, there exist differentiable functions $h_1, h_2$ and $h_3$ such that:
\[ x = h_1(y, z), \quad y = h_2(x, z), \quad z = h_3(x, y). \]
Using the equation for $y$ in the equation for $x$, and taking derivatives with respect to $x$ and $z$ we obtain:
\[ x = h_1(h_2(x, z), z) \Rightarrow 1 = \frac{\partial h_1}{\partial y} \frac{\partial h_2}{\partial x} \quad \text{and} \quad 0 = \frac{\partial h_1}{\partial y} \frac{\partial h_2}{\partial z} + \frac{\partial h_1}{\partial z}. \] (1)
Similarly, using the expression for $x$ in the equation for $z$ and taking derivatives with respect to $z$ and $y$ gives:
\[ z = h_3(h_1(y, z), y) \Rightarrow 1 = \frac{\partial h_3}{\partial x} \frac{\partial h_1}{\partial z} \quad \text{and} \quad 0 = \frac{\partial h_3}{\partial x} \frac{\partial h_1}{\partial y} + \frac{\partial h_3}{\partial y}. \] (2)
Combining the second portion of equation (1) and the first portion of equation (2) we obtain:
\[ 1 = \frac{\partial h_3}{\partial x} \frac{\partial h_1}{\partial z} = \frac{\partial h_3}{\partial x} \left( -\frac{\partial h_1}{\partial y} \frac{\partial h_2}{\partial z} \right). \]
Therefore:
\[ \frac{\partial x}{\partial y} \frac{\partial y}{\partial z} \frac{\partial z}{\partial x} = -1. \]
In the case of the ideal gas equation $pv = RT$, notice that:
\[ \frac{\partial p}{\partial v} = -\frac{RT}{v^2}, \quad \frac{\partial v}{\partial T} = \frac{R}{p}, \quad \frac{\partial T}{\partial p} = \frac{v}{R}. \]
Therefore:
\[ \frac{\partial p}{\partial v} \frac{\partial v}{\partial T} \frac{\partial T}{\partial p} = -\frac{RT}{v^2} \frac{R}{p} \frac{v}{R} = -1. \]
Q 4. (10 points) Let \( f : \mathbb{R}^3_x \to \mathbb{R}^3_y \) and \( g : \mathbb{R}^3_y \to \mathbb{R}^3_x \) be \( C^1 \) inverse functions. Show that:

\[
\frac{\partial g_1}{\partial y_1} = \frac{1}{J} \frac{\partial (f_2, f_3)}{\partial (x_2, x_3)}, \quad J = \frac{\partial (f_1, f_2, f_3)}{\partial (x_1, x_2, x_3)}.
\]

Obtain expressions for all other partial derivatives of \( g_1 \), and the corresponding formulas for the other components of \( g \).

Solution: For \( f \) and \( g \) mutual inverses, it follows that:

\[
f(g(y)) = y \quad \text{and} \quad g(f(x)) = x
\]

for any \( x, y \in \mathbb{R}^3 \). Therefore \( df_{g(y)} \circ dg_y = \text{Id} \) and \( dg_{f(x)} \circ df_x = \text{Id} \), where \( \text{Id} \) is the identity map. Thus:

\[
f'(x)g'(y) = I \quad \text{and} \quad g'(y)f'(x) = I,
\]

and the matrices \( f' \) and \( g' \) are the inverse of each other. Hence:

\[
g'(y) = \begin{bmatrix}
g_{11} & g_{12} & g_{13} \\
g_{21} & g_{22} & g_{23} \\
g_{31} & g_{32} & g_{33}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_3}{\partial x_1} \\
\frac{\partial f_1}{\partial x_2} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_3}{\partial x_2} \\
\frac{\partial f_1}{\partial x_3} & \frac{\partial f_2}{\partial x_3} & \frac{\partial f_3}{\partial x_3}
\end{bmatrix}^{-1}
= (f'(x))^{-1}.
\]

Let \( g'_i \) denote the \( i \)-th column of \( g' \). Then \( g'_i \) is the solution to the system \( f'(x)g'_i = e_i \), with \( e_i \) the \( i \)-th unit vector in the standard basis of \( \mathbb{R}^3 \). Using Cramer’s rule, with \( J = \det f'(x) \), we obtain:

\[
\frac{1}{J} \begin{vmatrix} 1 & \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_3} \\ 0 & \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_3} \\ 0 & \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_3}
\end{vmatrix}
= \frac{1}{J} \frac{\partial (f_2, f_3)}{\partial (x_2, x_3)}.
\]

Similarly:

\[
\frac{1}{J} \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & 1 & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_1}{\partial x_2} & 0 & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_1}{\partial x_3} & 0 & \frac{\partial f_3}{\partial x_3}
\end{vmatrix}
= \frac{1}{J} \frac{\partial (f_2, f_3)}{\partial (x_3, x_1)}.
\]

\[4\]
and
\[
\partial_1 g_3 = \frac{1}{J} \left| \begin{array}{ccc}
\partial_1 f_1 & \partial_2 f_1 & 1 \\
\partial_1 f_2 & \partial_2 f_2 & 0 \\
\partial_1 f_3 & \partial_2 f_3 & 0 \\
\end{array} \right| = \frac{1}{J} \frac{\partial (f_2, f_3)}{\partial (x_1, x_2)}
\]

The remaining elements of $g'$ are given by:

\[
\partial_2 g_1 = \frac{1}{J} \frac{\partial (f_3, f_1)}{\partial (x_2, x_3)}, \quad \partial_2 g_2 = \frac{1}{J} \frac{\partial (f_3, f_1)}{\partial (x_3, x_1)}, \quad \partial_2 g_3 = \frac{1}{J} \frac{\partial (f_3, f_1)}{\partial (x_1, x_2)},
\]

\[
\partial_3 g_1 = \frac{1}{J} \frac{\partial (f_1, f_2)}{\partial (x_2, x_3)}, \quad \partial_3 g_2 = \frac{1}{J} \frac{\partial (f_1, f_2)}{\partial (x_3, x_1)}, \quad \partial_3 g_3 = \frac{1}{J} \frac{\partial (f_1, f_2)}{\partial (x_1, x_2)}.
\]
Q 5. (10 points) If $U$ is an open subset of $\mathbb{R}^2_{uv}$ and $\varphi : U \to \mathbb{R}^3$ is a $C^1$ mapping, show that $\varphi$ is regular if and only if $\partial\varphi/\partial u \times \partial\varphi/\partial v \neq 0$ at each point of $U$. Conclude that $\varphi$ is regular if and only if, at each point of $U$, at least one of the Jacobian determinants

$$\frac{\partial(\varphi_1, \varphi_2)}{\partial(u, v)}, \quad \frac{\partial(\varphi_1, \varphi_3)}{\partial(u, v)}, \quad \frac{\partial(\varphi_2, \varphi_3)}{\partial(u, v)}$$

is nonzero.

Solution: The map $\varphi$ is regular $\iff$ for any $(u, v) \in U$ the rank of $\varphi'$ is 2 $\iff$ the columns of $\varphi'$, $\partial_u \varphi$ and $\partial_v \varphi$ are linearly independent. For two vectors $x_1, x_2 \in \mathbb{R}^3$, the cross product $x_1 \times x_2 = 0$ if and only if the two vectors are linearly dependent. Hence, $\partial_u \varphi \times \partial_v \varphi \neq 0 \iff \partial_u \varphi$ and $\partial_v \varphi$ are linearly independent $\iff \varphi$ is regular.

Moreover, $\partial_u \varphi \times \partial_v \varphi$ can be computed as:

$$\partial_u \varphi \times \partial_v \varphi = \begin{vmatrix} e_1 & e_2 & e_3 \\ \partial_u \varphi_1 & \partial_u \varphi_2 & \partial_u \varphi_3 \\ \partial_v \varphi_1 & \partial_v \varphi_2 & \partial_v \varphi_3 \end{vmatrix} = \frac{\partial(\varphi_2, \varphi_3)}{\partial(u, v)} e_1 + \frac{\partial(\varphi_3, \varphi_1)}{\partial(u, v)} e_2 + \frac{\partial(\varphi_1, \varphi_2)}{\partial(u, v)} e_3,$$

where $e_1, e_2$ and $e_3$ are the unit vectors of the standard basis of $\mathbb{R}^3$. That is, $\varphi$ is regular $\iff \partial_u \varphi \times \partial_v \varphi \neq 0 \iff$ at least one the aforementioned determinants is nonzero.