Q 1. (10 points) Given \( a \in \mathbb{R}^n \), denote by \( L_a \) the linear function
\[
L_a(x) = a \cdot x = \sum_{i=1}^{n} a_i x_i.
\]
Consider the norms of \( L_a \) with respect to the sup norm \( || \cdot ||_\infty \) and the 1-norm \( || \cdot ||_1 \) on \( \mathbb{R}^n \). Prove that \( ||L_a||_1 = |a|_\infty \), while \( ||L_a||_\infty = |a|_1 \).

Q 2. (10 points)

(a) Show that the linear mapping \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is one-to-one if and only if
\[
a = \max_{x \in \partial B_1} |T(x)|_\infty \text{ is positive.}
\]

(b) Conclude that the linear mapping \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is one-to-one if and only if there exists \( a > 0 \) such that \( |T(x)|_\infty \geq a|x|_\infty \) for all \( x \in \mathbb{R}^n \).

Q 3. (10 points) Suppose that the equation \( f(x, y, z) = 0 \) can be solved for each of the three variables \( x, y, z \) as a differentiable function of the other two. Prove that
\[
\frac{\partial x}{\partial y} \frac{\partial y}{\partial z} \frac{\partial z}{\partial x} = -1.
\]
Verify this in the case of the ideal gas equation \( pv = RT \), where \( p, v, T \) are the variables and \( R \) is a constant.

Q 4. (10 points) Let \( f : \mathbb{R}_x^3 \rightarrow \mathbb{R}_y^3 \) and \( g : \mathbb{R}_y^3 \rightarrow \mathbb{R}_x^3 \) be \( C^1 \) inverse functions. Show that:
\[
\frac{\partial g_1}{\partial y_1} = \frac{1}{J} \frac{\partial (f_2, f_3)}{\partial (x_2, x_3)}, \quad J = \frac{\partial (f_1, f_2, f_3)}{\partial (x_1, x_2, x_3)}.
\]
Obtain expressions for all other partial derivatives of \( g_1 \), and the corresponding formulas for the other components of \( g \).
Q 5. (10 points) If $U$ is an open subset of $\mathbb{R}^2_{uv}$ and $\varphi : U \to \mathbb{R}^3$ is a $C^1$ mapping, show that $\varphi$ is regular if and only if $\partial \varphi/\partial u \times \partial \varphi/\partial v \neq 0$ at each point of $U$. Conclude that $\varphi$ is regular if and only if, at each point of $U$, at least one of the Jacobian determinants

$$\frac{\partial (\varphi_1, \varphi_2)}{\partial (u, v)}, \quad \frac{\partial (\varphi_1, \varphi_3)}{\partial (u, v)}, \quad \frac{\partial (\varphi_2, \varphi_3)}{\partial (u, v)}$$

is nonzero.