Math 452 - Advanced Calculus II

Riemann Integration

1 Volume and the $n$-dimensional Integral

Definition 1. A closed interval in $\mathbb{R}^n$ is a set $I = I_1 \times I_2 \times \ldots \times I_n$, where $I_j = [a_j, b_j] \subset \mathbb{R}$. The volume of $I$ is $v(I) = (b_1 - a_1)(b_2 - a_2) \cdots (b_n - a_n)$.

Definition 2. A bounded subset $A$ of $\mathbb{R}^n$ is said to be contented if and only if there exists a non-negative real number, denoted $v(A)$, such that for any $\varepsilon > 0$, there exist:

(a) non-overlapping closed intervals $I_1, \ldots, I_p \subset A$ such that

$$\sum_{i=1}^{p} v(I_i) > v(A) - \varepsilon,$$

(b) closed intervals $J_1, \ldots, J_q$ such that

$$A \subset \bigcup_{i=1}^{q} J_i \quad \text{and} \quad \sum_{i=1}^{q} v(J_i) < v(A) + \varepsilon$$

The real number $v(A)$ is called the volume of $A$.

Definition 3. The contented set $A$ is said to be negligible if and only if $v(A) = 0$. In other words, $A$ is negligible if and only if given $\varepsilon > 0$, there exists a finite collection of intervals $\{J_j\}_{j=1}^{q}$ such that

$$A \subset \bigcup_{j=1}^{q} J_j \quad \text{and} \quad \sum_{j=1}^{q} v(J_j) < \varepsilon$$

Theorem 1. The bounded set $A$ is contented if and only if its boundary is negligible.

Proof. On the blackboard. \qed
Corollary 1. The intersection, union, or difference of two contented sets is contented.

Definition 4. Given a non-negative function $f : \mathbb{R}^n \to \mathbb{R}$, the ordinate set is defined as

$$O_f = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : 0 \leq x_{n+1} \leq f(x_1, \ldots, x_n)\}$$

Notice that the ordinate set of a function is bounded only if the function is bounded and has bounded support.

Definition 5. Given $f : \mathbb{R}^n \to \mathbb{R}$, the positive and negative parts $f^+$ and $f^-$ of $f$ are defined as:

$$f^+(x) = \max\{0, f(x)\} \quad \text{and} \quad f^-(x) = \max\{0, -f(x)\}.$$ 

In this case, $f = f^+ - f^-$. 

Definition 6. A bounded function $f$ with bounded support is said to be integrable if and only if the ordinate sets $O_{f^+}$ and $O_{f^-}$ are both contented. In this case

$$\int f = v(O_{f^+}) - v(O_{f^-}).$$

Definition 7. The characteristic function $\varphi_A$ of a set $A$ is defined as:

$$\varphi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

The integral of a function $f$ over the set $A$ is defined as

$$\int_A f = \int f \varphi_A$$

provided that the product $f \varphi_A$ is integrable.

Lemma 1. If the set $A$ is contented, then

$$\int \varphi_A = v(A)$$
Definition 8. The function $f$ is called admissible if and only if:

(a) $f$ is bounded,

(b) $f$ has bounded support,

(c) $f$ is continuous except on a negligible set.

Theorem 2. Every admissible function is integrable.

Proof. On the blackboard. □

- If $f : A \rightarrow \mathbb{R}$ is continuous and $A \subset \mathbb{R}^n$ is contended, then the graph of $f$ is a negligible set in $\mathbb{R}^{n+1}$.

- If $f$ is admissible and $A$ is contended, then $f \varphi_A$ is admissible.

- If the admissible function $f$ satisfies $|f(x)| \leq M$ for $x \in A$ with $A$ a contended set in $\mathbb{R}^n$, then

$$\left| \int_A f \right| \leq M v(A).$$

- If $A$ and $B$ are contended sets with $A \cap B$ negligible, and $f$ is admissible, then

$$\int_{A \cup B} f = \int_A f + \int_B f.$$

- Let $A$ be contended, and suppose the admissible functions $f$ and $g$ agree except on the negligible set $D$. Then

$$\int_A f = \int_A g.$$
2 Step Functions and Riemann Sums

Definition 9. A function \( h : \mathbb{R}^n \to \mathbb{R} \) is called a \textbf{step function} if and only if \( h \) can be written as a linear combination

\[
h = \sum_{i=1}^{p} a_i \varphi_i
\]

of characteristic functions \( \varphi_1, \ldots, \varphi_p \) of intervals \( I_1, \ldots, I_p \) whose interiors are mutually disjoint.

Theorem 3. If \( h \) is a step function then it is integrable, with

\[
\int h = \sum_{i=1}^{p} a_i v(I_i).
\]

Theorem 4. If \( h \) and \( k \) are step functions and \( c \in \mathbb{R} \), then \( ch \) and \( h + k \) are step functions and

\[
\int (h + k) = \int h + \int k.
\]

Theorem 5. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a bounded function with bounded support. Then \( f \) is integrable if and only if, given \( \varepsilon > 0 \), there exist step functions \( h \) and \( k \) such that

\[
h \leq f \leq k \quad \text{and} \quad \int (k - h) < \varepsilon,
\]

in which case \( \int h \leq \int f \leq \int k \).

Proof. On the blackboard.

Theorem 6. The set of integrable functions is a vector space. Moreover, for \( f, g \) integrable and \( \alpha, \beta \in \mathbb{R} \):

\[
\int (\alpha f + \beta g) = \alpha \int f + \beta \int g
\]

Proof. On the blackboard.
Recall that a partition of the interval $Q$ is a collection $\mathcal{P} = \{Q_1, \ldots, Q_k\}$ of closed intervals with disjoint interiors such that $Q = \bigcup_{i=1}^{k} Q_i$.

**Definition 10.** The mesh of a partition $\mathcal{P}$ is the maximum of the diameters of the intervals $Q_i$ in $\mathcal{P}$. A selection for $\mathcal{P}$ is a set $\mathcal{S} = \{x_1, \ldots, x_k\}$ such that $x_i \in Q_i$ for each $i$.

**Definition 11.** For $f : \mathbb{R}^n \to \mathbb{R}$ a function such that $f = 0$ outside of the interval $Q$, the Riemann sum for $f$ corresponding to the partition $\mathcal{P}$ and selection $\mathcal{S}$ is

$$R(f, \mathcal{P}, \mathcal{S}) = \sum_{i=1}^{k} f(x_i)v(Q_i).$$

**Theorem 7.** Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is bounded and vanishes outside the interval $Q$. If $f$ is integrable then, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left| \int f - R(f, \mathcal{P}, \mathcal{S}) \right| < \varepsilon$$

for $\mathcal{P}$ a partition of $Q$ with mesh $< \delta$, and $\mathcal{S}$ a selection of $\mathcal{P}$.

**Proof.** On the blackboard.