Math 452 - Advanced Calculus II
Iterated Integrals and Change of Variables

1 Iterated Integrals

**Theorem 1** (Fubini’s). Let $f : \mathbb{R}^{m+n} \to \mathbb{R}$ be an integrable function such that, for each $\mathbf{x} \in \mathbb{R}^m$, the function $f_\mathbf{x} : \mathbb{R}^n \to \mathbb{R}$, defined by $f_\mathbf{x}(\mathbf{y}) = f(\mathbf{x}, \mathbf{y})$, is integrable. Given contented sets $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$, let $F : \mathbb{R}^m \to \mathbb{R}$ be defined by

$$F(\mathbf{x}) = \int_B f_\mathbf{x} = \int_B f(\mathbf{x}, \mathbf{y}) \, d\mathbf{y}.$$ 

Then $F$ is integrable, and

$$\int_{A \times B} f = \int_A F.$$ 

**Proof.** On the blackboard. \(\square\)

**Theorem 2** (Cavalieri’s Principle). Let $A$ be a contented subset of $\mathbb{R}^{n+1}$, with $A \subset R \times [a, b]$ for $R \subset \mathbb{R}^n$ an interval. Suppose

$$A_t = \{ \mathbf{x} \in \mathbb{R}^n : (\mathbf{x}, t) \in A \} \subset \mathbb{R}^n$$

is contented for each $t \in [a, b]$, and write $A(t) = v(A_t)$. Then

$$v(A) = \int_a^b A(t) \, dt.$$ 

**Theorem 3.** If $A \subset \mathbb{R}^n$ is contented, and $f_1$ and $f_2$ are continuous functions on $A$ such that $f_1 \leq f_2$, then

$$C = \{ (\mathbf{x}, y) \in \mathbb{R}^{n+1} : \mathbf{x} \in A \text{ and } f_1(\mathbf{x}) \leq y \leq f_2(\mathbf{x}) \}$$

is a contented set. For $g : C \to \mathbb{R}$ a continuous function, it follows that

$$\int_C g = \int_A \left( \int_{f_1(\mathbf{x})}^{f_2(\mathbf{x})} g(\mathbf{x}, y) \, dy \right) \, d\mathbf{x}. $$
If $f : Q \to \mathbb{R}$ with $Q = [a_1, b_1] \times \cdots \times [a_n, b_n]$ an interval in $\mathbb{R}^n$, iterated applications of Fubini’s theorem give:

$$\int_Q f = \int_{a_n}^{b_n} \cdots \left( \int_{a_1}^{b_1} f(x) \, dx_1 \right) \cdots \, dx_{n-1} \, dx_n$$

If $Q$ is not an interval, it may be possible, using a suitable change of variables, to transform $\int_Q f$ into an integral of the form $\int_R g$, with $R$ an interval, and evaluate $\int g$ by use of Fubini’s theorem.

## 2 Change of Variables

The main idea is to use the local approximation of the transformation $T$ that maps the interval $Q$ into $A = T(Q)$ to compute

$$\int_A f = \int_{T(Q)} f = \int_Q g$$

for $g$ a suitable function.

**Theorem 4.** If $\lambda : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map, and $B \subset \mathbb{R}^n$ is contented, then $\lambda(B)$ is contented, and

$$v(\lambda(B)) = |\det \lambda| \cdot v(B).$$

**Proof.** On the blackboard.

**Corollary 1.** If $A \subset \mathbb{R}^n$ is a contented set and $\rho : \mathbb{R}^n \to \mathbb{R}^n$ is a rigid motion, then $\rho(A)$ is contented and $v(\rho(A)) = v(A)$.

**Lemma 1.** Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a $C^1$ map such that $f(0) = 0$ and $df_0 = I$. Suppose also that:

$$||df_x - I|| \leq \varepsilon \quad \forall x \in C_r = \{x \in \mathbb{R}^n : |x|_\infty \leq r\}.$$

Then $C_{(1-\varepsilon)r} \subset f(C_r) \subset C_{(1+\varepsilon)r}$.
Theorem 5. Let $Q$ be an interval centered at the point $a \in \mathbb{R}^n$, and suppose that $T : U \to \mathbb{R}^n$ is a $C^1$-invertible mapping on a neighborhood $U$ of $Q$. If there exists $\varepsilon \in (0, 1)$ such that

$$||dT_a^{-1} \circ dT_x - I|| \leq \varepsilon$$

for all $x \in Q$, then $T(Q)$ is contained with

$$(1 - \varepsilon)^n |\det T'(a)| v(Q) \leq v(T(Q)) \leq (1 + \varepsilon)^n |\det T'(a)| v(Q).$$

Proof. On the blackboard. \hfill \Box

Theorem 6 (Change of Variables). Let $Q$ be an interval in $\mathbb{R}^n$, and $T : \mathbb{R}^n \to \mathbb{R}^n$ a $C^1$-invertible mapping on a neighborhood of $Q$. If $f : \mathbb{R}^n \to \mathbb{R}$ is an integrable function such that $f \circ T$ is also integrable, then

$$\int_{T(Q)} f = \int_Q (f \circ T) |\det T'|.$$

Proof. On the blackboard. \hfill \Box

NOTE: The conclusion of the Change of Variables formula still holds if the map $T : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to be invertible on the interior of the interval $Q$ (instead of on a neighborhood of $Q$).
3 Improper Integrals and Absolutely Integrable Functions

Definition 1. The function \( f : U \rightarrow \mathbb{R} \) is **locally integrable** on \( U \) if and only if \( f \) is integrable on every compact contented subset of \( U \).

Definition 2. Let \( f : U \rightarrow \mathbb{R} \) be a locally integrable function on the open set \( U \subset \mathbb{R}^n \). It is said that \( f \) is **absolutely integrable** on \( U \) if and only if, given \( \varepsilon > 0 \), there exists a compact contented subset \( B_\varepsilon \) of \( U \) such that

\[
\left| \int_A f - \int_{B_\varepsilon} f \right| < \varepsilon
\]

for every compact contented set \( A \) satisfying \( B_\varepsilon \subset A \subset U \).

NOTE: The set of absolutely integrable functions forms a vector space.

Proposition 1. If the function \( f : U \rightarrow \mathbb{R} \) is absolutely integrable, then so is its absolute value \(|f|\).

Proof. On the blackboard. \( \square \)

Definition 3. The sequence \( \{A_k\}_{k=1}^\infty \) of compact contented subsets of \( U \) is called an **approximating sequence** for \( U \) if \( A_k \subset A_{k+1} \) for each \( k \geq 1 \) and \( U = \bigcup_{k=1}^\infty \text{int} \ A_k \)

Theorem 7. Suppose \( f \) is absolutely integrable on the open set \( U \subset \mathbb{R}^n \). Then there exists a number \( I = I_U f \) with the property that, given \( \varepsilon > 0 \), there exists a compact contented set \( C_\varepsilon \) such that

\[
\left| I - \int_A f \right| < \varepsilon
\]

for every compact contented subset \( A \) satisfying \( C_\varepsilon \subset A \subset U \). Moreover

\[
I = I_U f = \lim_{k \to \infty} \int_{A_k} f
\]

for every approximating sequence \( \{A_k\}_{1}^\infty \) for \( U \).
Proof. On the blackboard.

**Theorem 8.** If $U \subset \mathbb{R}^n$ is open and $f : U \rightarrow \mathbb{R}$ is integrable, then $f$ is absolutely integrable and

$$I_U f = \int_U f$$

Proof. On the blackboard.

**Definition 4.** The improper integral $\int_U f$ of a nonnegative function $f$ is said to be bounded if and only if there exists $M > 0$ such that

$$\int_A f \leq M$$

for every compact contented subset $A$ of $U$.

**Theorem 9.** Suppose that the nonnegative function $f$ is locally integrable on the open set $U$. Then $f$ is absolutely integrable on $U$ if and only if $\int_U f$ is bounded, in which case $\int_U f$ is the least upper bound of the values $\int_A f$, for all compact contented sets $A \subset U$.

Proof. On the blackboard.

**Corollary 2.** Suppose the nonnegative function $f$ is locally integrable on the open set $U$, and let $\{A_k\}_{1}^{\infty}$ be an approximating sequence for $U$. Then $f$ is absolutely integrable with

$$\int_U f = \lim_{k \rightarrow \infty} \int_{A_k} f,$$

provided that the limit exists (and is finite).

**Corollary 3.** Let $f : U \rightarrow \mathbb{R}$ be locally integrable. If $|f|$ is absolutely integrable on $U$, the so is $f$.

**Corollary 4** (Comparison Test). Suppose that $f$ and $g$ are locally integrable on $U$ with $0 \leq f \leq g$. If $g$ is absolutely integrable on $U$, then so is $f$. 
For real-valued functions of a single variable, we want to compare the improper integral denoted by
\[ \int_a^b f(x) \, dx, \]
defined for specific approximating sequences for \((a, b)\), with the improper integral
\[ \int_{(a,b)} f(x) \, dx, \]
defined if \(f\) is absolutely integrable (the same limit is obtained for all possible choices of approximating sequences).

**Theorem 10.** Suppose \( f : (a, b) \to \mathbb{R} \) is locally integrable with \( f \geq 0 \). Then \( f \) is absolutely integrable if and only if \( \int_a^b f \) converges, in which case
\[ \int_{(a,b)} f = \int_a^b f. \]