Polytopes, Combinatorics and Simplicial Complex

Chang Hai Bin

Supervisor: Wu Jie

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A “simple” result – Kruskal-Katona Theorem

Application to Simplicial Complexes

Introduction

Notation

We use \( (X_k) = \{ F \subseteq X : |F| = k \} \) to denote the collections of subsets of \( X \) with \( k \) elements.

For example, if \( X = \{ a, b, c, d \} \), then

\[
\binom{X}{3} = \left\{ \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\} \right\}
\]

\[
\binom{\mathbb{N}}{3} = \left\{ \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 5\}, \ldots \right\}
\]
Definition

For any $\mathcal{G} \subseteq 2^X$, we define the shadow of $\mathcal{G}$ as

$$\partial(\mathcal{G}) = \{ F - \{x\} : \text{for some } F \in \mathcal{G}, x \in F \}.$$
Example

For example,

\[ G = \left\{ \{1, 2, 3\} \right\} \]

then

\[ \partial(G) = \left\{ \{1, 2, 3\} - \{1\}, \{1, 2, 3\} - \{2\}, \{1, 2, 3\} - \{3\} \right\} \]

\[ = \left\{ \{1, 2\}, \{2, 3\}, \{1, 3\} \right\} \]
Example

For example,

\[ G_1 = \left\{ \{1, 2, 3\}, \{4, 5, 6\} \right\} \]

then

\[ \partial(G_1) = \left\{ \{1, 2\}, \{2, 3\}, \{1, 3\}, \{4, 5\}, \{5, 6\}, \{4, 6\} \right\} \]
Example

Another example,

\[ G_2 = \left\{ \{1, 2, 3\}, \{2, 3, 4\} \right\} \]

then

\[ \partial(G_2) = \left\{ \{1, 2\}, \{2, 3\}, \{1, 3\}, \{2, 4\}, \{3, 4\} \right\} \]
Example

\[ \partial G_1 = \{12, 13, 23, 45, 46, 56\}, \quad |\partial G_1| = 6 \]

\[ \partial G_2 = \{12, 13, 23, 24, 34\}, \quad |\partial G_2| = 5 \]
Example of Shadows

If we use 123 to denote the set \(\{1, 2, 3\}\), then

\[
\mathcal{G} = \{123, 134, 145, 125, 236, 346, 456, 256\}
\]

\[
\partial(\mathcal{G}) = \{12, 13, 14, 15, 23, 34, 45, 25, 26, 36, 46, 56\}
\]
Example

e.g. Let $X = \{a, b, c, d\}$, if $\mathcal{G} = \binom{X}{3} = \{abc, abd, acd, bcd\}$, then

$$\partial \mathcal{G} = \binom{X}{2} = \{ab, ac, ad, bc, bd, cd\}$$

In particular,

$$\partial \binom{X}{k} = \binom{X}{k-1}.$$
Minimize shadow

For $G \subseteq \binom{\mathbb{N}}{k}$, $|G| = m$, can we find a lower bound for the size of shadow (depending on $m, k$)?

$$|\partial G| \geq m^k$$
Proposition

For \( a, i \in \mathbb{Z}^+ \), we can find \( a_i > a_{i-1} > \cdots > a_j \geq j \geq 1 \), such that

\[
a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j},
\]

and the expression is unique. We call the following expression the \( i \)-cascade form of \( a \).

Proof.

Use greedy algorithm and induction, choose \( a_i \) such that

\[
\binom{a_i}{i} \leq a < \binom{a_i + 1}{i}
\]

and similar choices for \( a_{i-1}, \ldots, a_j \). \( \square \)
Definition

Given $i$-cascade form $a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j}$, $a_i > a_{i-1} > \cdots > a_j \geq j \geq 1$, we define

$$a[i] = \binom{a_i}{i-1} + \binom{a_{i-1}}{(i-1)-1} + \cdots + \binom{a_j}{j-1}, \quad 0[i] = 0$$
Kruskal-Katona Theorem

Theorem (Kruskal-Katona Theorem)

If \( G \subseteq \binom{\mathbb{N}}{k} \), \( |G| = m = \binom{a_k}{k} + \cdots + \binom{a_s}{s} \) (k-cascade form), then

\[
|\partial G| \geq m^{[k]} = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \cdots + \binom{a_s}{s-1}.
\]

Intuitively, the Kruskal-Katona Theorem means:

If we want to choose \( m \) elements from \( \binom{\mathbb{N}}{k} \) with the smallest shadow size, try not to introduce additional vertices / edges/ faces unless necessary.
Illustration

\[ G_1 = \{123, 456\}, \quad \partial G_1 = \{12, 13, 23, 45, 46, 56\} \]

\[ G_2 = \{123, 234\}, \quad \partial G_2 = \{12, 13, 23, 24, 34\} \]
Intuition behind Kruskal-Katona Theorem

How to rephrase “try not to introduce additional vertices unless necessary” in set theory/mathematically?

Answer: Like reading a dictionary.

321, 421, 431, 432, 521, 531, 532, 541, 542, 543, 621, ... 

In my report, I used reverse-lexicographic order, which is easier to visualize (but functions exactly the same as above).
Proof of Kruskal-Katona Theorem

The following shifting operation is first explored by Erdős, Ko, and Rado.

Definition

For \( x, y \in \mathbb{N}, x \neq y, \mathcal{G} \subseteq 2^{\mathbb{N}} \), define

\[
S_{x,y,\mathcal{G}}(H) = \begin{cases} 
(H - \{x\}) \cup \{y\} & \text{if } x \in H, y \notin H, \left[ (H - \{x\}) \cup \{y\} \right] \notin \mathcal{G} \\
H & \text{otherwise}
\end{cases}
\]

\[
S_{x,y,\mathcal{G}}(\mathcal{G}) = \{ S_{x,y,\mathcal{G}}(H) : H \in \mathcal{G} \},
\]

What the above operation does: Remove vertex \( \{x\} \) and replace by \( \{y\} \) whenever “possible”.

Polytopes, Combinatorics and Simplicial Complexes
Shifting Operation

A "simple" result – Kruskal-Katona Theorem

Application to Simplicial Complexes

Polytopes, Combinatorics and Simplicial Complexes

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Proof of Kruskal-Katona Theorem

Some recent proofs for this theorem.

1. P. Frankl (1984) used some clever observation to extract information out of a “flower-shaped” family.

2. Dömötör and Baljó (2005) used “large-set shifting” to reduce $\mathcal{G}$ to the best solution (smallest shadow size). Most intuitive proof in my opinion.
Proof of Kruskal-Katona Theorem

Other proofs for this theorem.

1. Proof by R. Ahlswede et all. using “Right-Left Shifting” (2003),
Abstract Simplicial Complexes

Definition

An abstract **simplicial complex** on the set \( S = \{1, 2, \ldots, m\} \) is a collection \( K = \{\sigma\} \) of subsets of \( S \) such that for each \( \sigma \in K \) all subsets of \( \sigma \) (including \( \emptyset \)) also belong to \( K \).

Intuitively, a simplicial complex is an attachment of \( n \)-dimensional triangles.
Simplicial Complex

In general, a simplicial complex could look something like this.
Abstract Simplicial Complex (Example)

Example: Tetrahedron attached to a triangle.

\[ K = \{\emptyset, \{1\}, \ldots, \{6\}, \{1, 2\}, \{1, 3\}, \ldots, \{1, 5\}, \{1, 6\}, \{5, 6\}, \{1, 2, 3\}, \{1, 2, 4\}, \ldots, \{1, 5, 6\}, \{1, 2, 3, 4\}\} \]
Abstract Simplicial Complex (Example)

The cube in $\mathbb{R}^3$ is NOT a simplicial complex.

\begin{itemize}
  \item e.g. $\{1, 2, 3, 4\} \in K$, but $\{1, 3, 4\} \notin K$.
\end{itemize}
If we let $F_i$ denote the number of $i$-elements subset of a simplicial complex $\Delta$, then using Kruskal-Katona Theorem,

**Theorem**

$(F_1, \ldots, F_d) \in \mathbb{N}^d$ correspond to some $(d - 1)$-dimensional simplicial complex if and only if

$$0 < F_{i+1}^{[i+1]} \leq F_i, \quad 1 \leq i \leq d - 1.$$
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A similar question

Definition

A monomial in the variables \( \{x_1, x_2, \ldots, x_\ell\} \) is a (monic) polynomial with only one term \( x_1^{a_1} \cdots x_\ell^{a_\ell} \), where \( a_i \geq 0 \ \forall i \).

We use \( M_k[x_1, x_2, \ldots] \) to denote the collection of monomials of degree \( k \) in the variables \( x_1, x_2, \ldots \).

For example,

\[
M_2[x, y, z] = \{x^2, y^2, z^2, xy, yz, xz\}.
\]

\[
M_3[x, y, z] = \{x^3, y^3, z^3, x^2y, x^2z, y^2x, y^2z, z^2x, z^2y, xyz\}
\]

We will use \( M_k \) to denote \( M_k[x_1, x_2, \ldots] \).
Shadow

Definition

For $G \subseteq M_k$, define

$$\partial G = \{ u \in M_{k-1} : \text{for some } v \in G, u | v, \deg(u) = k - 1 \}$$

For example, let $G = \{ x^3, x^2y, y^2z, z^3 \}$, then

$$\partial G = \{ x^2, xy, yz, y^2, z^2 \} .$$

e.g. If

$$G = M_3[x, y] = \{ x^3, x^2y, xy^2, y^3 \} ,$$

then

$$\partial G = M_2[x, y] = \{ x^2, xy, y^2 \} .$$
Minimize shadow

For $\mathcal{G} \subseteq \mathcal{M}_k, |\mathcal{G}| = m$, can we find a lower bound for the size of shadow (depending on $m, k$)?

$$|\partial \mathcal{G}| \geq m^k$$
Definition

Given *i*-cascade form \( a = \binom{a_i}{i} + \binom{a_{i-1}}{i-1} + \cdots + \binom{a_j}{j} \), \( a_i > a_{i-1} > \cdots > a_j \geq j \geq 1 \), we define

\[
a^{(i)} = \binom{a_i + 1}{i+1} + \binom{a_{i-1} + 1}{i-1 + 1} + \cdots + \binom{a_j + 1}{j+1}, \quad 0^{(i)} = 0.
\]

\[
a^{\{i\}} = \binom{a_i - 1}{i-1} + \binom{a_{i-1} - 1}{i-1 - 1} + \cdots + \binom{a_j - 1}{j-1}, \quad 0^{\{i\}} = 0.
\]
Theorem (Macaulay Combinatorial Theorem)

For \( m, k \in \mathbb{Z}^+ \), \( \mathcal{G} \subseteq \mathcal{M}_k \), \( |\mathcal{G}| = m = \binom{a_k}{k} + \cdots + \binom{a_s}{s} \) (\( k \)-cascade form). Then

\[
\partial \mathcal{G} \geq m^{\{k\}} = \binom{a_k - 1}{k - 1} + \cdots + \binom{a_s - 1}{s - 1}.
\]

Similarly, the intuition behind this Theorem is:

If we want to choose \( m \) elements from \( \mathcal{M}_k \) with the smallest shadow size, try not to introduce additional variables unless necessary.
Proof of Macaulay Combinatorial Theorem

This theorem looks similar to Kruskal-Katona Theorem.

However, the proofs introduced earlier cannot be easily translated to prove Macaulay Combinatorial Theorem. (I struggled with it for one week).

On the other hand, the “compression” method used by Clements and Lindström (1969) to prove Macaulay Combinatorial Theorem can be modified to prove Kruskal-Katona Theorem.
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Definition

*Convex polytope is convex hull/convex combination of finite set of points in \( \mathbb{R}^n \).*

All points in this polygon is a convex combination of the vertices.

\[
\{ t_1 x_1 + \ldots + t_n x_n \mid t_i \geq 0, t_1 + \ldots + t_n = 1 \}
\]
Simplicial Polytope

Definition

A polytope with $m$ vertices in $\mathbb{R}^n (m > n)$ is simplicial if no $(n + 1)$ of these vertices lie on a common affine hyperplane.

e.g. in $\mathbb{R}^3$, $n = 3$, any randomly chosen 4 points do not 'usually' fall on a same plane.

If the polytope (e.g. box on the left) has vertices that are not “well-behaved”, then it is not simplicial.
Definition

For a simplicial polytope, let \( f_i \) be the number of \( i \)-dimensional faces of \( S \). 
\[ f(S) = (f_0, \ldots, f_{n-1}) \] 
is called the \textbf{f-vector} of \( S \).

\[
\begin{align*}
  f_0 &= 4 \\
  f_1 &= 6 \\
  f_2 &= 4
\end{align*}
\]


**Definition**

For simplicial polytope in $\mathbb{R}^n$, we define the **h-vector** $(h_0, h_1, \ldots, h_n)$ such that the following holds:

$$h_0 t^n + h_1 t^{n-1} + \ldots + h_n = (t - 1)^n + f_0(t - 1)^{n-1} + \ldots + f_{n-1}$$
h-vector

Example: Tetrahedron, \((f_0, f_1, f_2) = (4, 6, 4)\).

\[
h_0 t^3 + h_1 t^2 + h_2 t + h_3 = (t - 1)^3 + 4(t - 1)^2 + 6(t - 1) + 4
\]

\((h_0, h_1, h_2, h_3) = (1, 3, 3, 1)\)

Notice that the \(h\)-vector is symmetric.
h-vector

Example: Pentagonal Bipyramid, \((f_0, f_1, f_2) = (7, 15, 10)\).

\[ h_0 t^3 + h_1 t^2 + h_2 t + h_3 = (t - 1)^3 + 7(t - 1)^2 + 15(t - 1) + 10 \]

\[ (h_0, h_1, h_2, h_3) = (1, 4, 4, 1) \]

Notice that the h-vector is symmetric.

Also, you can move the middle 5 vertices so that they do not lie on a plane, without changing the number of edges/faces etc.
Dehn-Sommerville Relations

Theorem (Dehn-Sommerville Relations)

*For an n-dimensional simplicial polytope in* \( \mathbb{R}^n \)

\[ h_i = h_{n-i}, \quad i = 0, 1, \ldots, n \]

i.e. There are a lot of equations that governs the h-vector and f-vector of simplicial polytope (and hence, the number of i-dimensional “faces”)
Theorem (g-Theorem)

An integer vector \((f_0, f_1, \ldots, f_{n-1})\) is the f-vector of a simplicial \(n\)-polytope if and only if the corresponding sequence \((h_0, \ldots, h_n)\) satisfies the following:

1. \(h_i = h_{n-i}, \ i = 0, \ldots, n\) (Dehn-Sommerville Equations)
2. \(h_0 \leq h_1 \leq \ldots \leq h_{\left\lfloor \frac{n}{2} \right\rfloor}, \ i = 0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\)
3. \(h_0 = 1, \ h_{i+1} - h_i \leq (h_i - h_{i-1})^{\langle i \rangle}, \ i = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor - 1.\)

Proven by R. Stanley (necessity) and Billera and Lee (sufficiency) in 1980. The proof is way too complicated for me to digest.
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Definition

The **Stanley-Reisner Face Ring** of a simplicial complex $K$ on the vertex set $\{1, 2, \ldots, m\}$ is the quotient ring

$$\mathbb{F}(K) = \mathbb{F}[v_1, \ldots, v_m]/I_K,$$

where $I_K$ is the homogeneous ideal generated by all square-free monomial $v_\sigma = v_{i_1}v_{i_2}\ldots v_{i_s} (i_1 < \ldots < i_s)$ such that $\sigma = \{i_1, \ldots, i_s\}$ is not a simplex of $K$.

The ideal $I_K$ is called the **Stanley-Reisner ideal** of $K$. 
Stanley-Reisner Ring (Examples)

Example: Boundary of a triangle.

\[ K = \{ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\} \} \]

\[ \{1, 2, 3\} \notin K \]

\[ I_K = (v_1 v_2 v_3) \]

\[ \mathbb{F}(\Delta) = \mathbb{F}[v_1, v_2, v_3]/(v_1 v_2 v_3) \]
Stanley-Reisner Ring (Examples)

E.g. \( \{1, 2, 5\} \notin K, \{2, 4, 5\} \notin K \)
\( \{1, 2, 4\} \in K, \)
\( \{4, 3\} \notin K, \)
\( \{1, 2\} \in K, \)
\( \{1, 2, 3, 4\} \notin K \)
\( \vdots \)

\[ I_K = \left( v_1 v_2 v_5, v_2 v_4 v_5, v_3 v_4, v_1 v_2 v_3 v_4, \ldots \right) \]
Definition

Let \( M = M^0 \oplus M^1 \oplus \ldots \) be a graded \( \mathbb{F} \)-algebra (i.e. a \( \mathbb{F} \) vector space with some multiplication property). Then, the series

\[
F(M; t) = \sum_{i=0}^{\infty} (\dim_{\mathbb{F}} M^i) t^i
\]

is called the **Poincare Series** of \( M \).
Lemma

The Poincare series of $\mathbb{F}(P)$ for a simplicial complex $P$ (in particular, a simplicial polytope) satisfy:

$$F(\mathbb{F}(P); t) = \sum_{i=-1}^{n-1} \frac{f_i t^{i+1}}{(1 - t)^{i+1}} = \frac{h_0 + h_1 t + \ldots + h_n t^n}{(1 - t)^n}$$

where $(f_0, \ldots, f_{n-1})$ and $(h_0, \ldots, h_n)$ are the $f$- and $h$-vectors of $P$ respectively.
Some results

We will invoke the following result from commutative algebra to prove a result in the next slide:

The $h$-vector of a simplicial sphere correspond to the $h$-vector of a multicomplex $\Gamma$.

In the main reference of my project, the authors in that book introduced the concepts of Krull dimension, Regular sequence, and Cohen-Macaulay Algebra to prove the above statement.
Theorem (Upper Bound Theorem)

The $h$-vector $(h_0, \ldots, h_n)$ of a simplicial $(n-1)$-sphere $K^{n-1} \subset \mathbb{R}^n$ with $m$ vertices satisfies

$$h_i(K^{n-1}) \leq \binom{m - n + i - 1}{i}, 0 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor.$$ 

Using the result from the previous slide, the Upper Bound Theorem is a consequence of the Macaulay Combinatorial Theorem.

The Upper Bound Theorem, together with Lower Bound Theorem (omitted in this talk) are both proven near the 1970s, and motivated P. McMullen to conjecture the $g$-theorem. On the other hand, these two Theorems are consequence of the $g$-theorem.
Conclusion

After learning about basic properties regarding polytopes (in semester 1)

I use January/February to study a few easy-to-state theorems (Kruskal-Katona Theorem, Macaulay Combinatorial Theorem), which have deep consequences (Upper Bound Conjecture).

Other contributions from the student include some properties for easy shapes (e.g. simplex and $n$-dimensional cube), as well as Lemmas that are not proven in the main reference.
Change in Research Direction

The full proof of $g$-theorem by Stanley and Billera and Lee encompass a lot of topics, including homology (Algebraic Topology), Commutative Algebra, Introduction to Algebraic Geometry, Torus Action, and Combinatorics.

At my current level of understanding, I can only make good progress in the combinatorics direction, and by quoting results from Commutative Algebra and Homology (without proof), I can tackle a subcase/corollary of the $g$-theorem.


Other publications/papers on Kruskal Katona Theorem and Macaulay Combinatorial Theorem.