Motterm 2.

1) Consider the $3 \times 4$ matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

a) Determine the range $\text{R}(A)$ (recall that the range is the same as the image $\text{im}(A)$). Are the columns of $A$ linearly independent? Find a basis of $\text{R}(A)$.

Solution: $\text{R}(A)$ is a subspace spanned by the columns of $A$. Call these columns:

$$a_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad a_3 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad a_4 = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$$

Obviously, $\text{R}(A) \subseteq \mathbb{R}^3$ which is a 3 dimensional space, so these 4 columns cannot be linearly independent.

Indeed, $a_3 = -a_1 + 2a_2$.

However, $a_1, a_2, a_4$ are linearly independent because

$$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 + \alpha_4 a_4 = \begin{pmatrix} \alpha_1 + 2\alpha_2 + 4\alpha_3 \\ \alpha_2 + 3\alpha_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

only if $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

Thus, $\text{R}(A) = \mathbb{R}^3$ and $a_1, a_2, a_4$ is a basis.
6) Determine the null space (or kernel) of $A$. What is the dimension of this null space? Determine a basis of this null space.

Solution: From the rank-nullity theorem we know that

$$\dim N(A) = 4 - \dim R(A) = 4 - 3 = 1.$$ 

We already noted that $\mathbf{a}_3 = -2\mathbf{a}_1 + 2\mathbf{a}_2 \implies A \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} = 0 \implies N(A) = \text{span} \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} \right\}$ which has basis given by vector $\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}$.

2) Consider $m \times p$ matrix $A$ and $p \times n$ matrix $B$, such that the null space (kernel) of $A$ is the same as the range (image) of $B$. Determine $N(AB)$. Can you determine $AB$?

Solution: Take any $x \in \mathbb{R}^m$ and obtain that $ABx = 0$ because $Bx \in R(B) = N(A)$. In particular, if $x = e_1$,

$\implies AB \mathbf{e}_1 = 0 = \text{first column of } AB

Proceed similarly by letting $x = e_j$ to conclude that all columns of $AB$ are zero. \implies AB = 0 \text{ and } N(AB) = \mathbb{R}^m.$
3) Suppose \( v_1, v_2, v_3 \) are linearly independent vectors in \( \mathbb{R}^3 \).

Show that \( u_1 = v_1, \ u_2 = v_1 + 2v_2, \ u_3 = v_1 + 2v_2 + 4v_3 \) are also linearly independent.

Solution: Consider linear relation \( \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 = 0 \) or equivalently

\[
\alpha_1 v_1 + \alpha_2 (v_1 + 2v_2) + \alpha_3 (v_1 + 2v_2 + 4v_3) = 0
\]

\[
= (\alpha_1 + \alpha_2 + \alpha_3) v_1 + (2\alpha_2 + 2\alpha_3) v_2 + 4\alpha_3 v_3 = 0
\]

Since \( v_1, v_2, v_3 \) are linearly independent \( \Rightarrow \)

\[
\alpha_1 + \alpha_2 + \alpha_3 = 0
\]

\[
2\alpha_2 + 2\alpha_3 = 0
\]

\[
4\alpha_3 = 0
\]

\( \Rightarrow \) \( \alpha_1 = \alpha_2 = \alpha_3 = 0 \) \( \Rightarrow \) only trivial linear relations exist among \( u_1, u_2, u_3 \) \( \Rightarrow \) they are linearly independent.

4) Consider the basis \( B \) of \( \mathbb{R}^2 \) consisting of vectors

\( b_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \ b_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \).

Let \( x = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \in \mathbb{R}^2 \). Find the coordinates of \( x \) in the basis \( B \), i.e. \( [x]_B \).
\[ x = \alpha_1 b_1 + \alpha_2 b_2 \implies [x]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \]

To find \( \alpha_1, \alpha_2 \) we must solve \((b_1, b_2)\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = x\) or explicitly:

\[
\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \implies \alpha_1 = 2, \quad \alpha_2 = -1
\]

Now let \( T: \mathbb{R}^2 \to \mathbb{R}^2 \) such that \( T(x) = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} x \). To find the matrix repres of \( T \) in \( B \) basis we write:

for arbitrary \( x \in \mathbb{R}^2 \), \([x]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}\) we have

\[
T(x) = Ax = \alpha_1 Ab_1 + \alpha_2 Ab_2
\]

\[
T(b_1) = Ab_1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3b_1
\]

\[
T(b_2) = Ab_2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} = -b_2
\]

Thus \( T(x) = 3 \alpha_1 b_1 - \alpha_2 b_2 \), which means:

\[
[T(x)]_B = \begin{pmatrix} 3\alpha_1 \\ -\alpha_2 \end{pmatrix} = M \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \quad \text{for all} \ x, \quad [x]_B = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}
\]

Let \([x]_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\) \( \implies M \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} \) = first column of \( M \)

\([x]_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\) \( \implies M \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \) = second column of \( M \) \( \implies M = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \)