Problem 30:

\[ f(t) = \det \begin{pmatrix} 1 & 1 \\ \frac{1}{a} & \frac{1}{b} \end{pmatrix} \]

Direct calculation gives:

\[ f(t) = (b-a)t^2 + t(a^2 - b^2) + ab^2 - a^2b = \]

\[ = (b-a)\left[ t^2 - (a+b)t + ab \right] \]

Coefficient of \( t^2 \) is \( b-a \)

Obviously \( f(b) = f(a) = 0 \), which follows by substitution in (1). We should get \( f(a) = 0 \) because when \( t = a \), the last two columns of the matrix are equal \( \Rightarrow \) determinant = 0.

Similarly, when \( t = b \), the last two columns are identical \( \Rightarrow \) determinant = 0.

Since \( f(t) \) is a polynomial of degree 2 with roots \( a, b \) \( \Rightarrow \)

\[ f(t) = k(t-a)(t-b) \]

with \( k = b-a = \) coefficient of \( t^2 \).

The matrix is invertible for all \( t \neq a, b \) since their determinant \( \neq 0 \).
Problem 31: $A_n = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_m \\ \vdots & \vdots & \ddots & \vdots \\ a_0^m & a_1^m & \cdots & a_m^m \end{pmatrix}$

When $n = 1 \Rightarrow$

a) $A_1 = \begin{pmatrix} 1 & 1 \\ a_0 & a_1 \end{pmatrix} \Rightarrow \det A = a_1 - a_0$

b) We know that $\det A_{n-1} = \prod_{i>j} (a_i - a_j)$

Now define:

$f(t) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ a_0 & a_1 & \cdots & a_{n-1} + t \\ \vdots & \vdots & \ddots & \vdots \\ a_0^m & a_1^m & \cdots & a_{n-1}^m + t^m \end{pmatrix} = \det \begin{pmatrix} A_{n-1} \vdots t \\ \vdots & a_0^m & \cdots & a_{n-1}^m \end{pmatrix}$

Using the Laplace (cofactor) expansion down the $n^{th}$ column $\Rightarrow$

$f(t) = (-1)^{m+m} \det A_{n-1} + (-1)^{m-1} \det (A_{n-1}, n) + \cdots + (-1)^{m+1} \det (A, n)$

where $A_{n-1} = \text{matrix obtained from } A_n \text{ by crossing } i^{th} \text{ row and } i^{th} \text{ column}$

Since $A_{n-1}$ is independent of $t \Rightarrow f(t) = \text{polynomial of degree m}$.

Moreover, coeff. of $t^m$ is $\det A_{n-1} = \prod_{i>j} (a_i - a_j), \quad i,j = 1, \ldots, n-1$.
But we know that \( f(a_0) = f(a_1) = \ldots = f(a_{n-1}) = 0 \).

\[ f(t) = k \prod_{j=0}^{n-1} (t - a_j) \quad \text{and} \quad k = \det A_{n-1}. \]

Letting \( t = a_n \), we have:

\[ \det A_n = f(a_n) = \det A_{n-1} \cdot (a_n - a_0)(a_n - a_1) \ldots (a_n - a_{n-1}). \]

\[ = \prod_{i<j} (a_i - a_j), \quad i, j = 1, \ldots, n. \]

**Problem 32.** This is a Vandermonde matrix with

\[ a_0 = 1, \quad a_1 = 2, \quad a_2 = 3, \quad a_3 = 4, \quad a_4 = 5. \]

\[ \det A = (a_1 - a_0)(a_2 - a_0)(a_3 - a_0)(a_4 - a_0)(a_2 - a_1)(a_3 - a_1) \cdot (a_4 - a_3)(a_4 - a_2)(a_4 - a_1)(a_3 - a_2)(a_3 - a_1) \]

\[ = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 1 = 288. \]