1) \( X = \begin{pmatrix} x_{11} & 0 & 0 \\ x_{12} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \)

\( X^2 = \begin{pmatrix} x_{11} & 0 & 0 \\ x_{12} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \begin{pmatrix} x_{11} & 0 & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix} = \begin{pmatrix} x_{11}^2 & 0 & 0 \\ x_{12}x_{21} & x_{22}^2 & 0 \\ x_{31}x_{11} + x_{32}x_{21} + x_{33}x_{31} & x_{32}x_{12} + x_{33}x_{22} & x_{33}^2 \end{pmatrix} \)

\( X^3 = \begin{pmatrix} x_{11} & 0 & 0 \\ x_{12} & x_{22} & 0 \\ x_{31} & x_{32} & x_{33} \end{pmatrix} \begin{pmatrix} x_{11}^2 & 0 & 0 \\ x_{12}x_{21} & x_{22}^2 & 0 \\ x_{31}x_{11} + x_{32}x_{21} + x_{33}x_{31} & x_{32}x_{12} + x_{33}x_{22} & x_{33}^2 \end{pmatrix} = \begin{pmatrix} x_{11}^3 & 0 & 0 \\ x_{12}(x_{11}^2 + x_{22}^2) & x_{22}^3 & 0 \\ x_{31}x_{11}x_{11} + x_{32}x_{12}x_{21} + x_{33}x_{31}x_{31} & x_{32}x_{12}x_{22} + x_{33}x_{22}x_{22} & x_{33}^3 \end{pmatrix} \)

The 11 entry of \( X^3 \) is \( x_{11}^3 = 0 \) \( \Rightarrow \) \( x_{11} = 0 \)

1,2 entry is \( 0 \) \( \Rightarrow \)

1,3 entry is \( 0 \) \( \Rightarrow \) \( x_{11} = 0 \)

2,1 entry is \( x_{21}x_{11}^2 + x_{22}(x_{11}^2 + x_{22}^2) = x_{22}^3 = 0 \) \( \Rightarrow \) \( x_{22} = 0 \)

2,2 entry is \( x_{22} = 0 \) consistent with \( x_{22} = 0 \)

2,3 entry is \( 0 \)

3,1 entry is \( x_{31}x_{11}^2 + x_{32}(x_{11}^2 + x_{22}^2) + x_{33}(x_{31}x_{11} + x_{32}x_{22} + x_{33}^2) \)

\( \Rightarrow x_{33}(x_{32}x_{22} + x_{33}^2) = 0 \)

\( \text{null } x_{33} = x_{22} = 0 \) \( \text{null } x_{22} = 0 \)

3,2 entry is \( x_{32}x_{22}^2 + x_{33}(x_{32}x_{22} + x_{33}^2) = x_{33}^2 = 0 \)

3,3 entry is \( x_{33}^3 = 0 \) \( \Rightarrow \) \( x_{33} = 0 \)

Thus, for \( X^3 = 0 \) we need \( x_{11} = x_{22} = x_{33} = 0 \)
2) \( A \) is \( n \times m \). Let us write it row by row:

\[
A = \begin{pmatrix}
\mathbf{r}_1 \\
\mathbf{r}_2 \\
\vdots \\
\mathbf{r}_m
\end{pmatrix}
\]

\( \mathbf{r}_i \) = row vector in \( \mathbb{R}^m \), for \( i = 1, \ldots, m \).

We seek \( X \) \( n \times m \) so that \( AX = I_m \). Let us write \( X \) column by column:

\[
X = \begin{pmatrix}
\mathbf{c}_1 \\
\mathbf{c}_2 \\
\vdots \\
\mathbf{c}_m
\end{pmatrix}
\]

\( \mathbf{c}_j \) = column vector in \( \mathbb{R}^m \), for \( j = 1, \ldots, m \).

The \( ij \) entry of \( AX \) is:

\[
(AX)_{ij} = \mathbf{r}_i \cdot \mathbf{c}_j = 1 \quad \text{if} \quad i = j \quad \text{and} \quad 0 \quad \text{otherwise}
\]

Thus, each column \( \mathbf{c}_j \) solves linear system:

\[
\begin{pmatrix}
\mathbf{r}_1 \\
\mathbf{r}_2 \\
\vdots \\
\mathbf{r}_m
\end{pmatrix}
\begin{pmatrix}
\mathbf{c}_j
\end{pmatrix}
= \begin{pmatrix}
\mathbf{c}_j
\end{pmatrix}
\]

To find \( \mathbf{c}_j \), we need to solve \( A \mathbf{c}_j = \mathbf{c}_j \).

Since \( \text{rank}(A) = m \Rightarrow \text{this is a consistent system} \)

If \( m = m \Rightarrow \mathbf{c}_j = \text{unique solution of } A \mathbf{c}_j = \mathbf{c}_j \)

If \( m < m \Rightarrow \text{we have infinitely many solutions} \)

Thus, if \( m < m \), there are infinitely many matrices \( X \) so that \( AX = I_m \).
3) \[ y_1 = x_1 + 3x_2 + 3x_3 \]
\[ y_2 = x_1 + 4x_2 + 3x_3 \]
\[ y_3 = 2x_1 + 7x_2 + 12x_3 \]

\[
\begin{pmatrix}
1 & 3 & 3 & 1 & 0 & 0 \\
1 & 4 & 8 & 0 & 1 & 0 \\
2 & 7 & 12 & 0 & 0 & 1 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 3 & 3 & 1 & 0 & 0 \\
0 & 1 & 5 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & -1 & 1 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & -12 & 14 & -3 & 0 \\
0 & 1 & 5 & -1 & 1 & 0 \\
0 & 0 & 1 & -1 & -1 & 1 \\
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 0 & -8 & -15 & 12 \\
0 & 1 & 0 & 4 & 6 & -5 \\
0 & 0 & 1 & -1 & -1 & 1 \\
\end{pmatrix}
\Rightarrow A \text{ is invertible}

\[
A^{-1} = \begin{pmatrix}
-8 & -15 & 12 \\
4 & 6 & -5 \\
-1 & -1 & 1 \\
\end{pmatrix}
\]

4) \[ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \] \quad \text{det}(A) = ad-bc = 1 \quad \text{and}

\[
A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = A \quad \Rightarrow \quad b = c = 0 \quad a = d
\]

Thus, \( A \) must be either: \( A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) or \( A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \)

5) If \( A \) is \( m \times n \) and has two equal columns, we need to show \( \text{rank}(A) < n \) \quad \Rightarrow A \text{ not invertible.}
Say the equal columns are $i$ and $j$. For $A$ to be non-singular we need $Ax = y$ to have unique solution for any $y$.

Writing $A$ column by column

$$A = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix}$$

$A_2 = \text{column vector in } \mathbb{R}^n$ for all $2 = 1, \ldots, n$

and using matrix vector product:

$$Ax = x_1 a_1 + \ldots + x_i a_i + \ldots + x_j a_j + \ldots + x_n a_n,$$

where we assumed $i \neq j$. Then, we see that:

$$Ax = \sum_{2=1}^{n} x_2 A_2 + (x_i + x_j) a_i = y.$$  

This problem cannot have a unique solution because we have infinitely many $x_i, x_j$ such that $x_i + x_j = \text{constant}$. Thus, $A$ is not invertible.