HW 2 Solutions:
Section 1.3 problems: Edition 5: 30, 36, 44, 48, 64, 66, 68
Edition 4: 30, 36, 44, 48, 64, 66, 68
Section 2.3 problems: Edition 5: 18, 26 Edition 4: 20, 24

1.
\[ x = \begin{pmatrix} 5 \\ 3 \\ -9 \end{pmatrix}, \quad y = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \]

Since \( Ax = y \), we need a matrix \( A \) which is 3x3. If the matrix is to have rank 1, then the RREF(A) must have two rows of 0's. This means that the second and third rows of \( A \) should be multiples of the first row:

\[ A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \alpha a_{11} & \alpha a_{12} & \alpha a_{13} \\ \beta a_{11} & \beta a_{12} & \beta a_{13} \end{pmatrix} \quad \text{for some } \alpha, \beta \in \mathbb{R} \]

Assuming \( a_{11} \neq 0 \) => RREF(A) = 
\[ \begin{pmatrix} 1 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \]

From \( Ax = y \) =>

\[ a_{11} \cdot 5 + a_{12} \cdot 3 + a_{13} \cdot (-9) = 2 \]
\[ \alpha a_{11} \cdot 5 + \alpha a_{12} \cdot 3 + \alpha a_{13} \cdot (-9) = 0 \]

\[ \Rightarrow \alpha = 0 \text{ and } \beta = \frac{1}{2} \]

and

\[ 5a_{11} + 3a_{12} - 9a_{13} = 2 \]

This has infinitely solutions. For example, 
\[ a_{12} = 1, \quad a_{13} = 1, \quad a_{11} = \frac{8}{5} \]
Check: \[ A = \begin{pmatrix} \frac{8}{5} & 4 & 1 \\ 0 & 0 & 0 \\ \frac{4}{5} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}, \quad A \cdot x = \begin{pmatrix} \frac{8}{5} \cdot 5 + 4 \cdot 3 + 1 \cdot (-9) \\ 0 \\ \frac{4}{5} \cdot 5 + \frac{1}{2} \cdot 3 + \frac{1}{2} \cdot (-9) \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix} \]

(2) Using the definition of matrix vector multiplication, we have: \[ A \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} = \text{first column of } A \]
\[ A \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{5} \\ \frac{5}{6} \end{pmatrix} = \text{second column of } A \]
\[ A \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{7}{8} \\ \frac{9}{8} \end{pmatrix} = \text{third column of } A \]

\[ A = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \]

(3) \[ A \text{ is } m \times m, \quad m > n. \]
\[ A \cdot x = x_1 \begin{pmatrix} a_{11} \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{m2} \end{pmatrix} + \cdots + x_m \begin{pmatrix} a_{1m} \\ a_{mm} \end{pmatrix} \]

We must show there exists \( b \) such that \( A \cdot x \neq b \). That is, \( b \) is not in span of columns of \( A \).

Since \( \text{rank}(A) \leq m < n \Rightarrow \text{RREF}(A) \) must have at least \( n-m \) rows of 0's.
Thus, any \( b \in \mathbb{R}^n \) that gives non-\( RREF \) non-zero entries in the last row, gives an inconsistent problem.

(4) \( A \mathbf{x}_1 = b \)

a) If \( \mathbf{x}_h \) is such that \( A \mathbf{x}_h = 0 \) then:
\[
\mathbf{x}_i + \mathbf{x}_h \text{ satisfies } A(\mathbf{x}_i + \mathbf{x}_h) = A\mathbf{x}_i + A\mathbf{x}_h = b
\]

b) This is as we did in class: \( A(\mathbf{x}_i - \mathbf{x}_2) = A\mathbf{x}_i - A\mathbf{x}_2 = b \)

c) \( A \in \mathbb{R}^{2 \times 2} \)

- Any vector on the line, say \( \mathbf{x}_h \) as shown in red of \( A \mathbf{x}_h = 0 \). Note that \( \mathbb{R}^2 = \text{span} \{ \mathbf{x}_i, \mathbf{x}_h \} \) because any \( \mathbf{u} \in \mathbb{R}^2 \) can be written as \( \mathbf{u} = a_1 \mathbf{x}_i + a_2 \mathbf{x}_2 \)

- To see this, let \( \mathbf{M} = (\mathbf{x}_i, \mathbf{x}_2) = \text{matrix with columns} \mathbf{x}_i, \mathbf{x}_2 \). Look at \( \text{RREF}(\mathbf{M}) \) and see that \( \mathbf{M} \) has rank

Then, if \( \mathbf{x} = \text{solution of } A \mathbf{x} = b \) then write:
\[
\mathbf{x} = a_1 \mathbf{x}_i + a_2 \mathbf{x}_h \text{ because } \mathbf{x} \in \mathbb{R}^2 = \text{span} \{ \mathbf{x}_i, \mathbf{x}_h \}
\]
Ax = α₁Ax₁ + α²Ax₂

All solutions of Ax = b are x₁ + α₂x₂ for all α₂

The set v + c w for c ∈ [0, 1]

is the segment on line parallel to v, passing through w, of length c.

The set for αv + βw = 1 is its parallelogram of sides v, w.

u ∈ ℝ² such that u · v = u · w =>

u · (v - w) = 0 => u is on vectors orthogonal to v - w. => belong to line in ℝ² orthogonal to v - w.
\[ A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \] and seek all \( B \) s.t. \( AB = BA \).

Since \( A \) is \( 2 \times 2 \) and for product \( AB \) to make sense we need \( B \) to have 2 rows, and for \( BA \) to make sense we need \( B \) to have 2 columns so that \( B \) must be \( 2 \times 2 \). We write it as: \( B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \) and seek \( b_{11}, \ldots, b_{22} \) so that

\[
\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}
\]

Calculating the products:

\[
\begin{pmatrix}
2b_{11} + 3b_{21} & 2b_{12} + 3b_{22} \\
-3b_{11} + 2b_{21} & -3b_{12} + 2b_{22}
\end{pmatrix} =
\begin{pmatrix}
2b_{11} - 3b_{12} & 3b_{11} + 2b_{12} \\
2b_{21} - 3b_{22} & 3b_{21} + 2b_{22}
\end{pmatrix}
\]

Thus, we must have:

\[
\begin{align*}
2b_{11} + 3b_{21} &= 2b_{11} - 3b_{12} & \Rightarrow & \quad b_{21} = -b_{12} \\
2b_{12} + 3b_{22} &= 3b_{11} + 2b_{12} & \Rightarrow & \quad b_{22} = b_{11} \\
-3b_{11} + 2b_{21} &= 2b_{21} - 3b_{22} & \Rightarrow & \quad b_{21} = b_{11} \\
-3b_{12} + 2b_{22} &= 3b_{21} + 2b_{22} & \Rightarrow & \quad b_{12} = -b_{21}
\end{align*}
\]

The matrix \( B \) must be of form:

\[
B = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}
\] for any \( \alpha \) and \( \beta \).

Note that \( A \) is such a matrix for \( \alpha = 2, \beta = 3 \).
We proceed similarly for $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$. Reasoning as before, $B$ must be $3 \times 3$, so we seek $b_{ij}$ for $i = 1, 2, 3$ and $j = 1, 2, 3$ so that:

\[
\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}
\]

Explicitly:

\[
\begin{pmatrix} 2 b_{11} & 3 b_{12} & 4 b_{13} \\ 2 b_{21} & 3 b_{22} & 4 b_{23} \\ 2 b_{31} & 3 b_{32} & 4 b_{33} \end{pmatrix} = \begin{pmatrix} 2 b_{11} & 2 b_{12} & 2 b_{13} \\ 3 b_{21} & 3 b_{22} & 3 b_{23} \\ 4 b_{31} & 4 b_{32} & 4 b_{33} \end{pmatrix}
\]

$\Rightarrow$ must have $b_{ij} = 0$ for $i \neq j$ $\Rightarrow$ $B$ diagonal implies matrices commute with $A$. Only diagonal matrices commute with $A$. 