Homological algebra (Math 613): Problem set 2

Bhargav Bhatt

1. Let \( f : K \to L \) be a map of chain complex over an abelian category \( A \). Construct a homotopy-equivalence \( \text{cone}(L \to \text{cone}(f)) \simeq K[1] \).

2. Check that any acyclic chain complex \( K \in \text{Mod}_R \) with \( K_i \) free and 0 for \( i \ll 0 \) is split.

3. Show that the assumption \( K_i = 0 \) for \( i \ll 0 \) above can be dropped when \( R = \mathbb{Z} \).

4. Show that \( K \in \text{Cl}_\text{nl}(\text{Mod}_R) \) is a projective object if and only if \( K \) is split and acyclic with each \( K_i \) projective.

5. Show that the homotopy category \( K(\text{Ab}) \) of abelian groups is not abelian.

Fix a cocomplete category \( A \). An object \( X \in A \) is called compact if \( \text{Hom}(X, -) \) commutes with filtered colimits, i.e., the natural map \( \text{Hom}(X, \text{colim} Y_i) \leftarrow \text{colim} \text{Hom}(X, Y_i) \) is a bijection for any filtered system \( \{Y_i\} \) of objects of \( A \). Write \( A^c \subset A \) for the full subcategory of all compact objects. We say that \( A \) is compactly generated if all objects in \( A \) are filtered colimits of objects in \( A^c \).

6. Show that \( A^c \) is closed under finite colimits in \( A \).

7. For the following cocomplete categories \( A \), describe \( A^c \), and determine if the category compactly generated:
   (a) Sets.
   (b) Groups.
   (c) Rings.
   (d) Commutative rings.
   (e) Open subsets of a topological space \( X \) (with morphisms being inclusion).
   (f) \( \text{Mod}_R \) for a ring \( R \).
   (g) \( \text{Ab}^{\text{opp}} \).

8. Given a set of rings \( \{R_i\} \), let \( R = \prod_i R_i \). Describe the compact objects in \( \text{Mod}_R \) in terms of compact objects in each \( \text{Mod}_{R_i} \).

9. For any small category \( C \), let \( \text{Ind}(C) \) be the category \( \text{ind-objects} \) in \( C \), i.e., objects are diagrams \( \{A_i\} \), indexed by filtered categories \( I \), and maps are given by \( \text{Hom}(\{A_i\}, \{B_j\}) = \text{colim}_i \text{lim}_j \text{Hom}(A_i, B_j) \). Show that if \( A \) is a compactly generated cocomplete category, then \( \text{Ind}(A^c) \simeq A \).

10. Determine whether the following functors \( \text{Ab} \to \text{Ab} \) are exact, left exact, right exact, exact in the middle, or neither:
   (a) \( F_1(A) = A/2A \).
   (b) \( F_2(A) = \{ x \in A \mid 4 \cdot x = 0 \} \).
   (c) \( F_2 \circ F_1 \) and \( F_1 \circ F_2 \), with \( F_1 \) and \( F_2 \) as above.
   (d) \( F(A) = A \otimes B \) for a fixed abelian group \( B \).
   (e) \( F(A) = A^{\otimes n} \).
11. Let $A$ be an abelian category, fix $X, Y \in A$, and $n \geq 1$. A degree $n$ Yoneda extension of $X$ by $Y$ is an exact sequence

$$Z_\bullet := 0 \to Y \to Z_1 \to \cdots \to Z_n \to X \to 0.$$ 

A map $Z_\bullet \to Z'_\bullet$ of such extensions is a map of exact sequences which is the identity on the $Y$ and $X$ terms. Two such extensions $Z'_\bullet$ and $Z''_\bullet$ are declared to be equivalent if there are maps $Z'_\bullet \to Z_\bullet \to Z''_\bullet$ of extensions.

(a) Show that equivalence of extensions is an equivalence relation on the set of all degree $n$ Yoneda extensions of $X$ by $Y$. The quotient set is denoted $\text{Ext}_A^n(X, Y)$.

(b) Show that $\text{Ext}_A^n(X, Y)$ is covariantly functorial in $Y$, and contravariantly functorial in $X$ by considering pushouts and pullbacks of extensions.

(c) Show that there is a natural binary operation $+$ on $\text{Ext}_A^n(X, Y)$ given by setting $[Z_\bullet] + [Z'_\bullet]$ to be the degree $n$ extension obtained by taking the direct sum $W_\bullet := Z_\bullet \oplus Z'_\bullet$, which is an element in $\text{Ext}_A^n(X \oplus Y, Y \oplus Y)$, and composing with the “fold” map $Y \oplus Y \to Y$ and the diagonal map $X \to X \oplus X$.

(d) Let $e_{X,Y}$ be the degree $n$ extension obtained as follows: $Z_1 = Y$, $Z_n = X$ and $Z_i = 0$ for $i \neq 1, n$ if $n \geq 2$, and $Z_1 = X \oplus Y$ if $n = 1$ (and the maps are the obvious ones in both cases). Show that $e_{X,Y}$ is a unit for the operation $+$ defined above.

(e) By tweaking signs, show that $\text{Ext}_A^n(X, Y)$ is an abelian group under $+$.

(f) For $X, Y, W \in A$, and $m, n \in \mathbb{Z}_{\geq 0}$, construct a natural map $\text{Ext}_A^n(X, Y) \times \text{Ext}_A^m(Y, W) \to \text{Ext}_A^{n+m}(X, W)$ by splicing extensions together. Show that this operation is bilinear with respect to $+$, and associative.

(g) Now assume $A = \text{Mod}_R$. Show that $\text{Ext}_A^n(X, -) = 0$ for all $n \geq 1$ if and only if $X$ is projective. Dually, show that $\text{Ext}_A^n(-, Y) = 0$ for all $n \geq 1$ if and only if $Y$ is injective.

(h) Given a short exact sequence $0 \to X \to Y \to Z \to 0$ in $A$, and $W \in A$, construct a natural map $\text{Ext}_A^n(X, W) \to \text{Ext}_A^{n+1}(Z, W)$. Using this, show that the family $\{\text{Ext}_A^n(-, W)\}$, together with these “boundary” maps, gives a $\delta$-functor $A^{\text{opp}} \to A$.

(i) Now assume $A = \text{Ab}$. Calculate $\text{Ext}_A^n(X, \mathbb{Z})$ using the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$. 

(f) $F(A) =$ free abelian group on the set $A$.

(g) $F(A) = A_{\text{tors}}$.

(h) $F(A) = \text{Hom}(A_{\text{tors}}, \mathbb{Q}/\mathbb{Z})$.

(i) Fix a topological space $X$ and $n \in \mathbb{Z}_{\geq 0}$, and let $F(A) = H^n(X, A)$.

(j) Fix a manifold $X$ of dimension $n$, and let $F(A) = H^n(X, A)$.