Algebraic topology (Math 592): Problem set 2

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1. Consider the circle $S^1$.
   (a) Let $f : S^1 \to S^1$. Given $x \in S^1$, we obtain an induced map $f_* : \mathbb{Z} \to \mathbb{Z}$, where the first $\mathbb{Z}$ is $\pi_1(S^1, x)$, while the second $\mathbb{Z}$ is $\pi_1(S^1, f(x))$. This map is uniquely determined by the integer $f_x(1)$, called the degree $\deg(f)$. Show that this integer is independent of $x$.
   (b) Given two maps $f, g : S^1 \to S^1$, show that $\deg(g \circ f) = \deg(g) \cdot \deg(f)$.
   (c) Show that if $f : S^1 \to S^1$ is not surjective, then $\deg(f) = 0$.
   (d) Show that any $f : S^1 \to S^1$ with $\deg(f) \neq 1$ must have a fixed point.
   (e) Construct a surjective map $f : S^1 \to S^1$ with $\deg(f) = 0$.

2. Let $X_n = \mathbb{R}^n - \{0\}$, and let $i_n : X_n \to X_{n+1}$ be the inclusion allowing us view $X_n$ as a subspace of $X_{n+1}$ of vectors with the last co-ordinate 0. We point these spaces using $x_n := (1, 0, 0, \ldots, 0) \in X_n$.
   (a) For any loop $\alpha$ in $\pi_1(X_n, x_n)$, show that the induced loop $i_n \ast (\alpha) \in \pi_1(X_{n+1}, x_{n+1})$ is 0.
   (b) Let $X_\infty = \cup_n X_n$, given the colimit (or “weak”) topology. For any compact Hausdorff space $Y$, show that any map $Y \to X_\infty$ factors through some $X_n$.
   (c) Show that any map $S^n \to X_\infty$ is null-homotopic for any $n \geq 0$. Conclude that $\pi_1(X_\infty, x_\infty) = 0$, where $x_\infty = (1, 0, 0, 0, \ldots)$.

We will see later that $\pi_1(X_n, x_n) = 0$ for $n \geq 3$.

3. We prove that fundamental groups of topological groups are abelian.
   (a) Let $S$ be a set equipped with two binary operations $*$ and $\otimes$. Assume that there exist units $e_*$ and $e_\otimes$ for each operation, and that $*: S \times S \to S$ is a homomorphism with respect to $\otimes$, i.e., $(a \otimes b) * (c \otimes d) = (a * c) \otimes (b * d)$. Show that $*$ is $\otimes$, and both are commutative.
   (b) Let $G$ be a topological group with identity $e$. Show that $\pi_1(G, e)$ is abelian.

4. Let $D \subset \mathbb{R}^2$ be the closed unit disc with boundary $S^1$. Let $f : D \to D$ be a map such that $f|_{S^1} = \text{id}_{S^1}$. Show that $f$ is surjective.

5. We will check that all maps $S^n \to S^1$ are null-homotopic. Let $\exp : \mathbb{R} \to S^1$ be the exponential $t \mapsto e^{2\pi it}$.
   (a) Let $D^o \subset \mathbb{R}^n$ be the open unit disc. Imitating the proof of $\pi_1(S^1, 1) \simeq \mathbb{Z}$, show that any map $f : D^o \to S^1$ lifts to a map $g : D^o \to \mathbb{R}$. Classify all possible such lifts $g$ of a given $f$.
   (b) Let $(X, x)$ be a pointed space such that $X = U_1 \cup U_2$, where each $U_i$ is homeomorphic to an open unit disc in $\mathbb{R}^n$, and $U_1 \cap U_2$ is path-connected. Show that for any map $f : X \to S^1$ and fixed $t_0 \in \mathbb{R}$ such that $\exp(t_0) = f(x)$, there is a unique map $\tilde{f} : X \to \mathbb{R}$ such that $\tilde{f}(x) = t_0$, and $\exp \circ \tilde{f} = f$.
   (c) Show that any map $S^n \to S^1$ is nullhomotopic for $n \geq 2$.

6. Given pointed spaces $(X, x)$ and $(Y, y)$, show that $\pi_1(X, x) \times \pi_1(Y, y) \simeq \pi_1(X \times Y, (x, y))$.

7. We prove that $\pi_1(S^n) = 0$ for $n \geq 2$. Fix a point $x_0 \in S^n$ with antipode $\overline{x_0}$. Let $U = S^n - \{x_0\}$, and $V = S^n - \{\overline{x_0}\}$. 

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(a) Show that $U$ and $V$ are homeomorphic to $\mathbb{R}^n$, and $U \cap V$ is homeomorphic to $\mathbb{R}^n - \{0\}$.

Fix a loop $\alpha : I \to S^n$ based at $x_0$. For $m \in \mathbb{N}$, write $\alpha_j : I \to S^n$ to be loop corresponding to $\alpha|_{[j/m, (j+1)/m]}$ through the standard reparametrization $I \simeq [j/m, (j+1)/m]$, so we obtain “factorisation” $\alpha = \alpha_{m-1} \cdot \alpha_{m-2} \cdots \alpha_0$ of $\alpha$ as a composition of $m$ paths.

(b) Show that if $m \gg 0$, then each $\alpha_j$ has image contained entirely in one of $U$ or $V$.

(c) Show that after repeatedly applying the following operations — replace some $\alpha_i$ by a homotopic path, and compose adjacent paths in the factorisation $\alpha = \alpha_{m-1} \cdot \alpha_{m-2} \cdots \alpha_0$ of $\alpha$ — we obtain a loop $\beta$ homotopic to $\alpha$ and a factorisation $\beta = \beta_n \cdot \beta_{n-1} \cdots \beta_0$ where each $\beta_i$ is a path in $S^n$ with image contained in $V$.

(d) Show that $\alpha$ is null-homotopic, and conclude that $\pi_1(S^n, x_0) = 0$.

8. Assume there exists some space $(X, x)$ such that $\pi_1(X, x)$ is not abelian. Show that $\pi_1(S^1 \vee S^1, 1)$ is not abelian. Here recall that for pointed spaces $(X, x)$ and $(Y, y)$, we write $X \vee Y := (X \sqcup Y)/(x \sim y)$.