Algebraic topology (Math 592): Problem set 11

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1. Construct CW structures on, and calculate the CW homology for, the following spaces:
   (a) $\mathbb{R}P^n$.
   (b) $\mathbb{C}P^n$.
   (c) The Klein bottle
   (d) The compact oriented surface $\Sigma_g$ of genus $g$.

2. Let $X$ be a CW complex with $n$-skeleton $X^n$. Assume $X^0 = \{*\}$. Show that:
   (a) The inclusion $X^1 \to X$ induces a surjection on $\pi_1$.
   (b) The inclusion $X^1 \to X$ induces a bijection on $\pi_1$ for $i \geq 2$.

3. Let $K$ be a chain complex of vector spaces over a field $k$. Define $\chi(K) = \sum_i (-1)^i \dim_k(H_i(K))$, viewed as an element in $\mathbb{Z} \cup \{+\infty\}$.
   (a) Assume that $K$ has finite dimensional terms $K_i$, and that $K_i = 0$ for $i \gg 0$ or $i \ll 0$. Prove that $\chi(K) = \sum_i (-1)^i \dim_k(K_i)$.
   (b) Let $0 \to K \to L \to M \to 0$ be an exact sequence of chain complexes of vector spaces. Show that $\chi(L) = \chi(K) + \chi(M)$.

4. Let $K$ be a chain complex of abelian groups such that each $K_i = \mathbb{Z} \oplus n_i$ for integers $n_i$, and $n_i = 0$ for $i \gg 0$ or $i \ll 0$. Prove the following:
   $$\sum(-1)^i n_i = \sum(-1)^i \dim_{\mathbb{F}_p} H_i(K/p) = \sum(-1)^i \dim_{\mathbb{Q}} H_i(K \otimes \mathbb{Q}).$$

5. Let $X$ be a finite CW complex, and let $s_n$ be the number of $n$-cells in $X$. Prove the following formula:
   $$\sum(-1)^i s_i = \sum(-1)^i \dim_{\mathbb{F}_p} H_i(X, \mathbb{F}_p) = \sum(-1)^i \dim_{\mathbb{Q}} H_i(X, \mathbb{Q}).$$
   In particular, the quantity on the left does not depend on the CW structure of $X$; it is called the Euler-Characteristic $\chi(X)$ of $X$.

6. Let $\pi : Y \to X$ be a covering space of degree $d$ with $X$ and $Y$ are slsc path-connected spaces.
   (a) For any $n \geq 0$, show that the map $\text{Map}(\Delta^n, Y) \to \text{Map}(\Delta^n, X)$ induced by composing with $\pi$ is surjective, and that the fibre over any element of $\text{Map}(\Delta^n, X)$ has exactly $d$ elements.
   (b) Define a map $C_n(X) \to C_n(Y)$ of abelian groups by sending a simplex $\sigma \in \text{Map}(\Delta^n, X)$ to the sum of its preimages in $\text{Map}(\Delta^n, Y) \subset C_n(Y)$. Show that this defines a map $\pi^* : C_*(X) \to C_*(Y)$ of chain complexes such that $\pi_* \circ \pi^*$ is multiplication by $d$.
   (c) Generalize the construction of $\pi^*$ to $C_*(\cdot, k)$ for any ring $k$. Conclude that the induced map $\pi_* : H_i(Y, k) \to H_i(X, k)$ is surjective if $d$ is invertible on $k$.
   (d) Give an example of a map $\pi$ as above such that $H_i(Y, \mathbb{Z}/2) \to H_i(X, \mathbb{Z}/2)$ is not surjective.
   (e) Give an example of a covering space $f : Z \to W$ of slsc path-connected spaces (necessarily of infinite degree) such that $f_* : H_i(Y, \mathbb{Q}) \to H_i(X, \mathbb{Q})$ is not surjective.