1. Introduction

So in this talk I will give a brief review of zeta functions for prehomogeneous vector spaces and then look more at the case of binary cubic forms. In a sense these objects are generalisations of the Riemann Zeta Function. Recall the functional equation

$$
\zeta(1 - s) = (2\pi)^{-s} \Gamma(s)(2\cos \frac{\pi s}{2})\zeta(s)
$$

Recall the steps of the proof:

1. The following identity for the Schwartz class:

$$
\int_{\mathbb{R}^n} |x|^{s-1} \hat{\Phi}(x) dx = a(s) \int_{\mathbb{R}^n} |x|^{-s} \Phi(x) dx
$$

2. Poission Summation Formula applied to $Z(\hat{\Phi}, s)$ takes us to $Z(\hat{\Phi}, s)$.

The key idea here is that Fourier transform of $|x|^{s-1}$ as a distribution is again a complex power $|x|^{-s}$. In general if $P(x) = x_1^2 + \ldots + x_n^2$ we have that $|P|^{s-n/2} = a_p(s)|p|^{-s}$ and this gives the Functional equation of the Epstein Zeta function.

Here’s the problem: Find a polynomial $P$ such that $|\hat{P}|^s$ is again a complex power.

**Theorem 1.1** (M. Sato). A relative invariant polynomial $P$ of a prehomogeneous vector space is such a one!

**Definition 1.1.** $(G, V)$ is a finite dimensional algebraic representation of a connected linear algebraic group defined over $\mathbb{C}$ is a pre-homogeneous vector space (hereby denoted as $P.V$) if there exists $V' \subset V$ Zariski open orbit and the set $S = V \setminus V'$ is called the singular set.

$P \in \mathbb{C}[V]$ is a relative invariant is a relative invariant iff $P(g.x) = \chi(g)P(x)$ for some $\chi \in X^*(G)$. Now if $G$ is reductive and $S$ is an irreducibly hypersurface, the polynomial defining $S$ is a relative invariant.

If we take the dual representation of $G$ we also have a prehomogeneous vector space and $S^*$ is an irreducible hypersurface and its defining polynomial $P^*$ also turns out to be a relative invariant and the character thus obtained $\chi^* = \chi^{-1}$. Now if $(G, V)$ defined over $K \subset \mathbb{C}$, we can take $P \in K[V]$.

Ex: $G_m$ acting in the usual way having the invariant $x$, $O(n)$ having the invariant $P(x) = x_1^2 + \ldots + x_n^2$. See also Sato-Kimura, Bhargava.
2. Local Theory over $\mathbb{R}$

Assume that $(G,V)/\mathbb{R}$ and let $V_\mathbb{R} = V_1 \cup \ldots \cup V_\ell$ be orbit decomposition of the open orbit over $\mathbb{C}$ and similarly for the dual rep. Fact: $\ell = \ell^*$.

**Definition 2.1.** (*Local Zeta Function*) Let

$$\Gamma_i(\Phi,s) = \int_{V_i} |P(x)|^s \Phi(x) dx$$

for $\Phi \in \mathcal{S}(V_\mathbb{R})$ and

$$\Gamma^*_j(\Phi^*,s) = \int_{V^*_j} |P^*(y)|^s \Phi^*(y) dy$$

for $\Phi^* \in \mathcal{S}(V^*_\mathbb{R})$. Theory of $L$-function implies a meromorphic continuation.

Recall: If $\Phi \in C^\infty_c(V_i)$, then it turns out that $\Gamma_i(\Phi,s)$ is entire since $|P|$ is also bounded.

Now $G_\mathbb{R}$ acts on $\mathcal{S}(V_\mathbb{R})$ as $(g \cdot \Phi)(x) = \Phi(g^{-1}x)$. What happens for

$$\Gamma_i(g\Phi,s) = \int_{V_i} |P(x)|^s \Phi(g^{-1}x) dx$$

for $x \mapsto gx$? We get

$$\frac{d(gx)}{dx} |\chi(g)|^s \Gamma_i(\Phi,s) = |\chi(g)|^{s+n/d} \Gamma_i(\Phi,s)$$

Taking the usual Fourier transform takes us from $\mathcal{S}(V_\mathbb{R})$ to $\mathcal{S}(V^*_\mathbb{R})$ and we have $g \cdot \hat{\Phi} = |\chi(g)|^{n/d}(g \cdot \hat{\Phi})$ and we get

$$\Gamma^*_j(g \cdot \hat{\Phi},s - n/d) = |\chi(g)|^{-s+n/d} \Gamma^*_j(\hat{\Phi},s - n/d)$$

Recall that $V_i$ is $G_\mathbb{R}$-homogeneous and for $\Phi$ having compact support in $V_i$, the uniqueness of relatively invariant distributions implies that

$$\Gamma^*_j(\hat{\Phi}s - n/d) = c_{ji}(s) \Gamma_i(\Phi,-s)$$

**Theorem 2.1.** (*weak. L.F.E*) If $\Phi \in C^\infty_c(V_\mathbb{R})$, we have that $\Gamma^*_j(\hat{\Phi},s-n/d) = \sum_{1 \leq i \leq \ell} c_{ji}(s) \Gamma_i(\Phi,-s)$ by decomposing $\Phi$ into parts defined over the various real orbits.

Indeed Sato proved

**Theorem 2.2 (Sato).** This is true for all Schwartz functions.

(1) The same is true over a local field $F$, simple over $\mathbb{C}$ and for non-archimedean fields it’s more difficult (cf. Igusa zeta function).

(2) When $V_F$ is not a single $G_F$-orbit, the local uniqueness fails and this is why the zeta function doesn’t have an Euler product in most interesting cases.
3. Global Theory over $\mathbb{Q}$

Let $(G, V)/\mathbb{Q}$ and $\Gamma G \mathbb{Q}$ be an arithmetic subgroup and $V_\mathbb{Q}$ be a $\Gamma$-invariant lattice, $L^* \subset V_\mathbb{Q}^*$ is a dual lattice.

**Definition 3.1.**

$$Z(\Phi, L, s) = \int_{G_\mathbb{R}/V} |\chi(g)| \sum_{\chi \in L \setminus S_0} \Phi(gx) dg$$

is the zeta integral and

$$\zeta_i(L, s) = \sum_{\chi \in \Gamma \setminus L \cap V_i} \frac{\mu(x)}{|P(x)|^s}$$

is the zeta function.

**Lemma 3.1.** $\Phi \in C^\infty_c(V_i)$ implies that $Z(\Phi, L, s) = \zeta_i(L, s) \Gamma_i(\Phi, s - n/d)$ and similarly for $Z^*, \zeta^*_j$.

Assume finiteness of $\mu(x)$ for all $x \in L$.

**Theorem 3.2.** We have $Z(\Phi, L, s)Z^*(\Phi^*, L^*, s)$ converges absolutely for $\Re(s) >> 0$ and same for $\zeta_i \zeta^*_j$.

Analytic continuation and functional equation works out nicely using Fourier transform. In particular, picking a test function $\Phi$ such that $\Phi|_{S_\mathbb{R}} = 0$ and $\hat{\Phi}|_{S_\mathbb{R}} = 0$. Let’s divide $Z$ into two parts $Z_+$ and $Z_-$ (integrating over the parts where $\chi(g) \geq 1$ or $< 1$) and $Z_+$ turns out to be entire by absolute convergence for large enough $s$. For $Z_-$ we use the vanishing of $\Phi$ on the singular set and the Poisson summation formula to show entireness and we get

$$Z(\Phi, L, s) = \zeta_i(L, s) \Gamma_i(\Phi, s - n/d)$$

and further

$$Z(\Phi, L, n/d - s) = v(L)^{-1} Z^*(\hat{\Phi}, L^*, s)$$

**Theorem 3.3** (Sato-Shintani). We get $\zeta_i(L, n/d - s) = \sum_{1 \leq j \leq \ell} c_{ji} \zeta^*_j(L^*, s)$

Now, L.F.E and the above theorem give us that

$$Z(\Phi, L, n/d - s) = v(L)^{-1} Z^*(\hat{\Phi}, L^*, s)$$

is true for $\Phi \in \mathcal{S}(V_\mathbb{R})$.

4. Binary Cubic Forms

This zeta function was first written down by Shintani. Let $G := GL_2$ and $V := \text{Sym}^3(\mathbb{Q}^2)$ and let $G$ act on $V$ in the usual way by change of variables, scaled by determinant. It turns out that the discriminant is relatively invariant under the action. We choose $\Gamma := G_\mathbb{Z}$ and $L := V_\mathbb{Z}$ and there is some identification between $V_\mathbb{Q}$ and $V_\mathbb{Q}^*$ that restricts to the lattices. In this case there are two real orbits determined by the sign of the discriminant of the binary cubic form and the singular set has zero discriminant (degeneracy locus).

$$\zeta_\pm(s) := \sum_{x \in G_\mathbb{Z}\backslash V_\mathbb{Z}} \frac{|\text{Stab}(x; G_\mathbb{Z})|^{-1} s}{|P(x)|}$$

and this has an analytic continuation and a functional equation.
Theorem 4.1 (Delone-Faddeev, Gan-Gross-Savin). We have the following correspondence
\[ G\mathbb{Z}\setminus \mathbb{V}\mathbb{Z} \leftrightarrow \{R: \text{cubic ring}\}/\sim \]
which is discriminant preserving and further this correspondence preserves the stabilizer.
Therefore we can write \( \zeta_{\pm}(s) = \sum_{R \in A_3(\mathbb{Z})} \frac{|\text{Aut}(R)|}{|\text{Disc}(R)|^s}. \)

Primary issues in applying the Shintani Zeta Function to the global theory at present is the determination of poles and residues. We have \( \Gamma_+(s) \) to be generically entire and non-zero for appropriate choice of \( \Phi \) and Poisson summing up, it is enough to study the \( Z_- \) part since the others are entire. Now the singular set can be stratifies as double, triple roots and zero. The contribution from 0 is \( \frac{\tau(G)\Phi(0)}{s-1} \) but unfortunately the parts diverge. However, Shintani found cancellation in a clever way giving simple poles at 1 and \( 5/6 \) and holomorphic elsewhere. Applying Tauberian theorems we get that
\[ \sum_{R \in A_3(\mathbb{Z}), 0 \leq \pm D(X) < X} \frac{1}{\text{Aut}(R)} = r_1^{\pm}(X) + r_5^{\pm} \frac{x^{5/6}}{5/6} + O(x^{3/5+\epsilon}) \]
and applying prime sieve we can count maximal rings and cubic fields.

5. Generalisations to Number Fields

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