1. Introduction

Last week Manjul talked about pre-homogeneous spaces that had a single orbit over \( \mathbb{C} \) but
interesting properties over \( \mathbb{Z} \). Today, we’ll be interested in studying a more general class of repre-
sentations which can have moduli even over \( \mathbb{C} \) called **co-regular spaces**.

**Definition 1.** A coregular representation of an algebraic group \( G \) is a \( \mathbb{C} \)-representation such that
the ring of invariants of the semi-simple part is a free polynomial ring. The space \( V \) is said to be
a coregular space.

Some examples of coregular spaces were mentioned last time - binary quadratic forms, binary
cubic forms etc. Clearly, any pre-homogeneous space is a vector space. Coregular spaces were first
classified by Littelman (1989) for semisimple irreducible representations; there are around 50 of
them including infinite families. A lot of these spaces come from Vinberg theory, something that’ll
be discussed over the next couple of weeks, but there are others as well. In particular, several
coregular spaces are related to genus 1 curves as the title of this talk suggests. We are interested
in the \( K \)-orbits of these spaces, \( V_K / G_K \) where \( K \) is any field of characteristic different from 2, 3.
Our prototypical example will be \( \mathbb{Q} \). We see that

\[
G_K \backslash V_K \overset{1-1}{\leftrightarrow} \{ \text{Genus 1 curves + extra data} \}
\]

where the extra data takes the form of line bundles and more generally vector bundles. Other
coregular representations give higher genus curves, some of which Manjul and Dick Gross have
been studying.

2. Genus One Curves

Before we talk about genus one curves, let’s review the correspondences for **rings**.

- Cubic Rings – binary cubic forms, \( \text{Sym}^3(2) \)
- Quartic Rings – pairs of ternary quadratic forms, \( 2 \otimes \text{Sym}^2(3) \)
- Quintic Rings – Pfaffians, \( 4 \otimes \bigwedge^2 5 \)

In some sense, the spaces parametrising genus one curves are rather analogous to the rings
situation – the degree of the ring is replaced by the degree of the line bundle on the genus one
curve. We will find that in some sense, they’re **one step up** from genus one curves.

2.1. **Genus one curves with degree 3 line bundles.** These curves correspond to orbits of
ternary cubic forms under \( \text{PGL}(V) \times \mathbb{G}_m \) (this is essentially the \( \text{GL}(V) \) action but you’re modding
out base-change) i.e. \( \text{Sym}^3(V) / (\text{PGL}(V) \times \mathbb{G}_m) \). Dimensionally, this is a step up from binary
cubic forms which were \( \text{Sym}^3(2) \). A ternary cubic is a cubic polynomial in \( \mathbb{P}^2 \) which is genus 1 by
Riemann-Roch. If you pull back \( \mathcal{O}(1) \) to the cubic, you get a degree 3 line bundle (can be thought
of as the cubic curve intersecting a line in \( \mathbb{P}^2 \) at 3 points, by Bézout’s theorem, yielding a degree 3
divisor). We’re interested in nodegenerate orbits with non-zero discriminant since these correspond
to smooth cubics with specified degree 3 line bundles.

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2.2. **Genus one curves with degree 4 line bundles.** These curves turn out to correspond to orbits of pairs of quaternary quadratic forms under the action of GL(2) × GL(4) i.e. GL(2) × GL(4) \( \setminus \) 2 ⊗ Sym^2(4) which is a 'step up' again. Geometrically, we have two quadrics in \( \mathbb{P}^3 \) whose intersection is a genus 1 curve by Riemann-Roch and the Hilbert polynomial and thus give you a degree 4 line bundle. The nondegenerate orbits thus correspond to a curve with a specified degree 4 line bundle.

2.3. **Genus one curves with degree 5 line bundles.** These curves turn out to correspond to orbits of \( 5 \otimes \bigwedge^2 5 \) under the action of GL(5) × SL(5) which is once more a 'step up'. Geometrically, these correspond to the intersection of 5 Pfaffians in \( \mathbb{P}^4 \) which intersect in a genus 1 curve. Pulling back \( \mathcal{O}(1) \) from \( \mathbb{P}^4 \) gives a degree 5 line bundle and thus non-degenerate orbits correspond to pairs of genus 1 curves and a specified degree 5 line bundle.

2.4. **Genus one curves with degree 2 line bundles.** We're now ready to consider the degree 2 case. Any genus one curve effects a double cover of \( \mathbb{P}^1 \) branched at 4 points, by Riemann-Hurwitz. It turns out that the right objects to consider are binary quartic forms \( \text{Sym}^4(V) \) under \( \text{PGL}(V) \times \mathbb{G}_m \). Roughly speaking, you need to twist by discriminant twice to get the right action. Now, \( y^2 = \text{binary quartic form} \) corresponds to a smooth genus 1 curve which as a double cover of \( \mathbb{P}^1 \) has an associated degree 2 line bundle, as desired.

In each of these cases, it’s important that the invariant ring of the representation is generated by two elements freely. For the degree 2 case \( I, J \) generate the invariant ring and \( \deg J/\deg I = \frac{3}{2} \). We find that the Jacobian of the curve gives an elliptic curve \( E : y^2 = x^3 + Ix + J \). The line bundles are irrelevant over \( \mathbb{C} \) since they’re classified by degree alone and so \((I,J)\) determine the orbit over \( \mathbb{C} \).

3. **Rubik’s Cubes**

Now to continue in Manjul’s vein from last week, we’re now going to talk about Rubik’s cubes which are \( 3 \times 3 \times 3 \) cubes. Later we’ll briefly discuss hypercubes which are \( 2 \times 2 \times 2 \times 2 \) cubes. It turns out these are the basic spaces that we care about.

Now, if we take 3 slices \( M_1, M_2, M_3 \) of our Rubik’s cube in a specific direction, we get \( 3 \times 3 \) matrices \( M_1, M_2, M_3 \) which in turn give us a cubic \( \det(M_1x + M_2y + M_3z) \) which is a cubic in those three variables. Clearly we have a genus 1 curve in \( \mathbb{P}^2 \) and the pull-back of \( \mathcal{O}(1) \) gives a degree 3 line bundle \( L \). We could have done this in all 3 different directions and doing so gives us 3 genus 1 curves \( C_i \) with degree 3 line bundles \( L_i \). We look at the space of Rubik’s cubes \( V_1 \otimes V_2 \otimes V_3 \) upto \( \text{GL}_3^3 \) equivalence. It turns out unsurprisingly, that all three curves \( C_i \) are isomorphic. This follows since the matrix \( M_1x + M_2y + M_3z \) is zero, taking the kernel in some direction gives you a map from \( C_1 \) to \( C_2 \in \mathbb{P}(V_2^*) = \mathbb{P}^2 \) and doing this in the opposite direction gives you the reverse map. Smoothness conditions amount to the rank of this matrix being 2. Now composing the isomorphisms gives us a non-trivial automorphism of the individual curves:

\[
\begin{array}{ccc}
\pi_{31} & \pi_{12} & \pi_{23} \\
C_1 & C_2 & C_3 \\
\end{array}
\]

We pull back the line bundles to get curve \( C_1 \) and lines bundles \( L_1, L_2 = \pi_{12}^* L_2', L_3 = (\pi_{31}^{-1})^* L_3' \).

We find that \((C, L_1, L_2, L_3)\) is enough to characterise the orbits of the representation of \( \text{GL}_3^3 \).

**Theorem 3.1.** We have the following correspondence:

\[
K\text{-points of orbits} \leftrightarrow \{(C, L_1, L_2, L_3)|L_1^\otimes 2 = L_2 \otimes L_3\}
\]
Theorem 4.1. The reverse map is given via the multiplication map on line bundles. It is the kernel of \( H^0(C, L_1) \otimes H^0(C, L_2) \rightarrow H^0(C, L_1 \otimes L_2) \) which is a 3-dimensional subspace of \( 3 \times 3 \) matrices which yield a Rubik's cube. It turns out that the non-trivial automorphism that we saw above is tantamount to adding a point \( P \) on the Jacobian to \( C \), the point here being \( L_1 \otimes L_2^* \).

3.1. Invariant Theory of Cubes. We find that the generators of these invariants have degrees 6, 9 and 12 and the Jacobian can be written down as \( y + d_9y = x^3 + d_6x^2 + d_4x \) clearly having points \([0:0:1]\) and \([0:1:0]\) and is therefore an elliptic curve. On average, it turns out the Jacobian has non-zero rank, which in some sense is rather surprising and a contrast to the expected distribution.

Now, we're going to make the same construction more abstractly. Given the \( 3 \times 3 \times 3 \) of \( \mathbb{Z} \otimes \mathbb{Z}^4/\mathbb{GL}_4 \). By similar slicing constructions, we find that we get a \( 2 \times 2 \times 2 \) matrix of bi-degree (1, 1) forms whose determinant are bidegree (2, 2) polynomials which cut out a variety in \( \mathbb{P}^1 \times \mathbb{P}^1 \) which is generally a genus 1 curve. Alternatively, restrict to a double cover of \( \mathbb{P}^1 \) ramified at 4 points from which you get a binary quartic form which we already know corresponds to a genus 1 curve and some degree 2 line bundles. Eventually, it turns out that we have 4 degree 2 line bundles \( L_1, L_2, L_3, L_4 \) such that \( L_1 \otimes L_2 \cong L_3 \otimes L_4 \).

The stratification is done by rank; \( Y \) is the intersection of the image of \( \mathbb{P}(V_1^* \otimes V_2) \) in \( \mathbb{P}(V_1 \otimes V_2) \). For a \( 3 \times 3 \) matrix of rank 2, its adjugate has rank 1. This gives us a blowdown \( Y \rightarrow X \) which is defined precisely on \( Y \times X \). Looking at the image of \( C_1 \) on \( X \) under this map gives the curves \( C_2 \) and \( C_3 \).

The idea is to replace \( 3 \times 3 \) matrices with \( 3 \times 3 \) Hermitian matrices with respect to some quadratic algebra over \( K \). This allows us to replace \( X \) with eg. Severi varieties and their twists. We list here some correspondences of varieties \( X \) with the rep. theory world, \( Y \) being the secant variety (which coincidentally is the tangent variety).

- \( \mathbb{P}^2 \) embedded using the Veronese in \( \mathbb{P}(\text{Sym}^2 3) = \mathbb{P}^5 - \text{GL}_3 \times \text{GL}_3 \) acting on \( 3 \otimes \text{Sym}^2 3 \)
- \( \text{Gr}(2, 6) \) embedding using the Plucker in \( \mathbb{P}(\Lambda^2 6) = \mathbb{P}^{14} - \text{GL}_3 \times \text{GL}_6 \) acting on \( 3 \otimes \Lambda^2 (6) \)
- \( E_{16} \) which is exceptional – \( \text{GL}_2 \times E_6 \) acting on \( 3 \otimes 27 \).

4. Hypercubes

These are \( 2 \times 2 \times 2 \) cubes or in other words \( \mathbb{Z}^\otimes 4/\mathbb{GL}_4 \). By similar slicing constructions, we find that we get a \( 2 \times 2 \times 2 \) matrix of bi-degree (1, 1) forms whose determinant are bidegree (2, 2) polynomials which cut out a variety in \( \mathbb{P}^4 \times \mathbb{P}^4 \) which is generally a genus 1 curve. Alternatively, restrict to a double cover of \( \mathbb{P}^1 \) ramified at 4 points from which you get a binary quartic form which we already know corresponds to a genus 1 curve and some degree 2 line bundles. Eventually, it turns out that we have 4 degree 2 line bundles \( L_1, L_2, L_3, L_4 \) such that \( L_1 \otimes L_2 \cong L_3 \otimes L_4 \).

Theorem 4.1. Non-degenerate orbits of hypercubes correspond to genus one curves \( C \) with 4 line bundles such that \( L_1 \otimes L_2 \cong L_3 \otimes L_4 \).

Similar to the case of Rubik's cubes, we get isomorphisms between the \( C_{ij} \). If we fix \( i \) and vary \( j \), we get a 3-cycle which turns out to yield a hyperelliptic involution of \( C_{ij} \) while a four cycle yields a point on the Jacobian. In fact, there are 3 interesting 4-cycles and the points obtained on the Jacobian sum to zero!! The entire Hermitianization story can be carried out here as well, but it’s more complicated geometrically and has to do with the secant variety. The ring of invariants is polynomials generated by elements of degrees 2, 4, 4 and 6 and turns out to be coregular.

One final note – we can use this correspondence with the Jacobian and restrict to locally soluble curves to count Selmer groups. That’s all for today!