1. Generalities

Key focuses of this class have included algebras of rank $n$ and genus one curves. The key notion of what unites these two is the notion of a del Pezzo variety. Let $k$ be a field of characteristic 0.

**Theorem 1.1.** Let $X$ be an irreducible projective variety not contained in a hyperplane, then $\text{deg}(X) \geq \text{codim}(X) + 1$. When equality is attained, $X$ is said to have minimal degree.

**Theorem 1.2.** (ref. Eisenbud’s geometry of syzygies) There’s a classification of the above collection – some of these are quadrics, rational normal curves, ration normal scrolls.

**Definition 1.1.** Let $X \in \mathbb{P}^n$ be an irreducible non-singular variety such that $\text{deg}(X) = \text{codim}(X) + 2$; $X$ is said to be of almost minimal degree.

Let’s make a list of these things:

- cubics in $\mathbb{P}^2$ – genus one curves.
- complete intersection of two quadrics in $\mathbb{P}^3$ – genus one curves.

More generally, we have:

- deg 3: any cubic hypersurface.
- deg 4: complete intersection of 2 quadrics – $2 \otimes \text{Sym}^2 n$
- deg 5: Grassmanian $\text{Gr}(2, 5) \subset \mathbb{P}^2 5 = \mathbb{P}^9$ – which turn out to be cut out by $n \otimes \wedge^2 5$.
- deg 6: $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$ and $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$
- deg 7: blow up of $\mathbb{P}^3$ in a point.
- deg 8: $\mathbb{P}^3$ in its 2-uple embedding.

1.1. Alternative Characterizations.

**Theorem 1.3** (Adjunction Formula). For $X \subset Y$ of codimension 1, we have that $\wedge^{\dim X} \Omega^1 = K_X = (K_Y + X)|_X$

Working this out in the case of cubic hypersurfaces and complete intersections of 2 quadrics, we get $K_X = \mathcal{O}_{\mathbb{P}^n}(1 - \dim X)$.

**Definition 1.2.** A pair $(X, L)$ is del Pezzo if $X$ is a non-singular $k$-variety, $L$ is ample and $K_X = L^{\otimes (1 - \dim X)}$
More generally, we say that a pair is Fano of co-index $i$ if $K_X = L^{-(\dim X + 1 - i)}$ and there’s a theorem classifying Fano varieties of co-index 3.

We can add in:

- $\deg 1$: $\deg 6$ hypersurfaces in $\mathbb{P}(1, 2, 3)$ also turn down to be elliptic curves.
- $\deg 2$: Double cover of $\mathbb{P}^n$ ramified at a quartic which for $n = 1$ is just a genus one curve.

**Theorem 1.4.** Let $X \subset \mathbb{P}^n$ be a variety of almost minimal degree, then the minimal free resolution of $R/I_X$ where $R$ is the homogeneous coordinate ring of projective space and $I_X$ is the ideal cutting out $X$ inside the homogeneous coordinate ring, is pure and the $\beta_i$ correspond to dimensions of certain representations of $S_d$.

**Sketch.** $X$ del-Pezzo is on the one hand projectively normal and on the other hand, arithmetically Cohen-Macaulay $R/I_X$. You can check both these conditions by taking linear sections. This suggests that we may as well take linear sections which gives us $d$ points in $\mathbb{P}^{d-2} = \mathbb{P}V$ and we know that $\text{PGL}_{d-1}$ acts transitively on these sets and this contains an $S_d$ and it turns out there are two lifts of $S_d$ to $\text{GL}_{d-1}$ (Kevin here draws wonderful diagrams that I don’t reproduce) and these correspond to the quadrics cutting out $d$ points in $\mathbb{P}^{d-2}$ and everything works out very nicely. □

**Remark 1.5.** This theorem has a corollary stating that there exists an associative commutative differential graded structure on the minimal free resolution and in the case of 4, this is Koszul multiplication and in the case of 5 it’s Buchsbaum-Eisenbud multiplication.

**Remark 1.6.** Why can’t we count? Answer: This gives a homogeneous space whose integer points are rank $n$ rings (with resolvents) but it’s no longer a vector space. Degree 3 stuff is $\text{Sym}^3 n$, degree 4 stuff is $2 \otimes \text{Sym}^3 n$ and degree 5 stuff is $n \otimes \wedge^2 5$. However, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ gives 6 points in $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2 \times \mathbb{P}^2$ gives 6 points in $\mathbb{P}^2$.

2. **del Pezzo varieties**

If $X$ is a surface and you blow up in points and there’s an intersection pairing on curves which is the number of intersection points counted with multiplicity. Lines in a non-singular surface have self-intersection $-1$, if $X$ is non-singular all curves on $X$ have self-intersection at least $-1$. Exceptional divisors have self-intersection $-1$ and moreover all lines come from blowing up and you can blow down. If you blow up $\mathbb{P}^2$ in $d$ points the lines on the blow up are the exceptional divisors of the blow-up, the $\mathbb{P}^2$-lines through any points blown up, the conics through 5 points and cubics through all points with a double point.

**Theorem 2.1.** Every del Pezzo surface is the blow-up of $\mathbb{P}^2$ in $d$ points or $\mathbb{P}^1 \times \mathbb{P}^1$

Example: cubic surface $X$ having degree 3 has 27 lines as known famously.

Any pair of six lines $(\ell_1, \ldots, \ell_6)$ and $(m_1, \ldots, m_6)$ such that $(\ell_i, \ell_j) = (m_i, m_j) = -\delta_{ij}$ and $(\ell_i, m_i) = 0$ is called a double-sixer and the goal is to get $4 \times 3 \times 3$ boxes. So if you have 6 points $P$ sitting inside $\mathbb{P}^2$, $\dim S_p = 1 = \text{depth}(S_p) = \text{depth}(I_P)$ and by a theorem of Hilbert-Burch we get a $3 \times 4$ matrix of linear forms on $\mathbb{P}^2$.

**Theorem 2.2** (Segre). Non-degenerate orbits of $3 \times 3 \times 4$ boxes on the surface are non-singular cubic surfaces with a choice of rational double-sixers.
The degree 2 case: Instead of blowing up 6 points in $\mathbb{P}^2$, I blow up 6 points in $\mathbb{P}^1 \times \mathbb{P}^1$ this will turn out to be a degree 2 del Pezzo surface.

Start with a $2 \times 2 \times 2 \times 3$ box which gives me a $\text{Sym}^2 \otimes \text{Sym}^2 3$ and $X \subset \mathbb{P}^1 \times \mathbb{P}^2$ and taking discriminants of the forms we get a degree 4 polynomial on $\mathbb{P}^2$.

Those six points come from the $2 \times 2$ minors of this $3 \times 3$ matrix of $(1, 1)$ forms.

There are a whole host of theorems that arise in this fashion:

**Theorem 2.3.** Non-degenerate orbits of $2 \times 2 \times 2 \times 3$ boxes are in $1 - 1$ correspondence with azygetic complexes of bitangents on $C - (S_\alpha, S_\beta, S_\gamma) = S_\alpha$ is 6 pairs of bitangents with some properties.

**Theorem 2.4.** Non-degenerate orbits of $\text{Sym}^2 \otimes \text{Sym}^2 3$ are in correspondence with plane quartics with a point on the Jacobian

3. Conclusion

These representations are the natural higher dimensional analogues of the Vinberg representations ...