1. Background on K3 Surfaces

**Definition 1.1.** A K3 surface over a field \( k \) is a smooth, projective, geometrically integral surface \( X \) such that \( H^1(X, \mathcal{O}_X) = 0 \) and \( K_X \cong \mathcal{O}_X \) (trivial canonical bundle).

Over \( \mathbb{C} \), we can think of simply connected surfaces having a global non-vanishing holomorphic 2-form.

1.1. Examples.

1. Smooth quartic surface in \( \mathbb{P}^3 \) (\( K_{\mathbb{P}^3} \cong \mathcal{O}_{\mathbb{P}^3}(-4) \) and then adjunction formula). If A,D,E singularities, blow up is a K3 surface.

2. Smooth complete intersections of a quadric and cubic in \( \mathbb{P}^4 \).

3. Smooth complete intersections of three quadrics in \( \mathbb{P}^5 \).

4. Smooth zero-locus of a \((2,2,2)\) form on \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \).

5. Double cover of \( \mathbb{P}^2 \) branched along a smooth sextic.

1.2. Cohomology of K3 surfaces. Since \( H^1(X, \mathbb{Z}) = 0 \), we have to look at second cohomology. Turns out that \( H^2(X, \mathbb{Z}) \) is torsion free and is a free abelian group of rank 22 (by using Noether’s formula). \( (b_0 = 1, b_4 = 1) \) \( H^2(X, \mathbb{Z}) \) has a cup-product grading and is therefore a lattice and it turns out to be even (Wu’s formula). The matrix of this pairing has determinant \( \pm 1 \) by Poincare duality and signature \((3,19)\) by Hirzebruch’s index formula. Milnor tells us that \( H^2(X, \mathbb{Z}) \cong E_8(-1)^2 \oplus U^3 = II_{3,19} \).

We have \( H^2(X, \mathbb{Z}) \otimes \mathbb{C} \cong H^{2,0} \oplus H^{0,2} \oplus H^{1,1} \). Consider the exponential exact sequence

\[
0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X^\times \rightarrow 0
\]

and take the long exact sequence of cohomology. We have that the Chern class map \( H^1(X, \mathcal{O}_X^\times) := \text{Pic}(X) \hookrightarrow H^2(X, \mathbb{Z}) \); so linear equivalence is equivalent to algebraic equivalence. So \( \text{Pic}(X) \cong \text{NS}(X) \) is the Neron-Severi lattice and is a sublattice of \( H^2(X, \mathbb{Z}) \) contained in \( H^{1,1}(X, \mathbb{Z}) \cap H^2(X, \mathbb{Z}) \) having rank \( \rho \leq 20 \) and signature \((1, \rho, -1)\) (Hodge-index thm). It’s also primitive in \( H^2(X, \mathbb{Z}) \) and its quotient \( H^2(X, \mathbb{Z})/\text{Pic}(X) \) equals \( H^{1,1}(X, \mathbb{Z}) \cap H^2(X, \mathbb{Z}) \). The transcendental lattice \( T_X \) is the orthogonal complement of the algebraic divisors inside the cohomology.
Let \( H^{1,1}(X, \mathbb{R}) = \{ x \in H^2(X, \mathbb{R}) : \langle x, \omega_X \rangle = 0 \} \). The form \( \langle , \rangle \) has signature \((1,19)\) on \( H^{1,1}(X, \mathbb{R}) \) and \( \{ x \in H^{1,1}(X, \mathbb{R}) : \langle x, x \rangle > 0 \} \) has 2 connected components and exactly one of these contains ample divisors (or Kahler classes). Call this the positive cone.

**Theorem 1.1** (Torelli). Let \( X \) and \( X' \) be K3 surfaces and suppose we’re given an isometry \( \eta : H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z}) \) which sends \( H^2,0(X, \mathbb{C}) \) to \( H^2,0(X', \mathbb{C}) \) and also sends the positive cone of \( X' \) to the positive cone of \( X \) and finally we need that it sends effective disors to effective divisors. Then \( \eta = \phi^* \) for a unique isomorphism \( \phi : X \rightarrow X' \).

We have a period mapping from marked K3 surfaces \((X, \phi)\) where \( \phi \) is a marking from \( H^2(X, \mathbb{Z}) \) to a fixed lattice \( \Lambda \), to \( \Omega = \{ \omega \in \Lambda \otimes \mathbb{C} : \langle \omega, \omega \rangle > 0 \}/\mathbb{C}^\times \) given by taking \((X, \phi)\) to \( \phi(\omega_X) \). Note that if \( \Omega \) is an open subset of a quadric in \( \mathbb{P}^{21} \) and if the K3 surface is algebraic, say \( z \in \Lambda \) such that \( z^2 > 0 \) and looking at moduli space of K3 surfaces \((X, \phi)\) such that \( \phi^{-1}(z) \) is the class of an ample line bundle on \( X \), the image of the period map lies inside \( \Omega_z = \{ \omega \in \Omega : \langle \omega, z \rangle = 0 \} \) and this thing is 19-dimensional.

Original Torelli theorem about algebraic K3 surfaces (Piatettski-Shapiro-Shafarevich) showed that the period map was injective and Kulikov showed surjectivity for algebraic K3 surfaces. Burns and Rapoport showed Torelli for arbitrary K3 surfaces and Todorov showed surjectivity.

### 1.3. Lattice polarized K3 surfaces (Nikodim).

Suppose that \( M \in \Lambda \) is some primitive sublattice of signature \((1, x)\), we can look at marked K3 surfaces \((X, \phi)\) such that \( \phi^{-1}(M) \) is contained in the \( NS(X) \). Let \( N = M^\perp \) in \( \Lambda \) and
\[
\Omega_M = \{ \omega \in N \otimes \mathbb{C} : \langle \omega, \omega \rangle = 0, \langle \omega, \overline{\omega} \rangle > 0 \}/\mathbb{C}^\times
\]
and it turns out that this thing is two copies of a Hermitian symmetric domain. Let
\[
\Gamma(M) = \{ \sigma \in O(\Lambda) : \sigma \text{ fixes } M \text{ pointwise} \}
\]
and let \( \Gamma_M \) be the image of \( \Gamma(M) \) in \( O(N) \) and then get an isomorphism
\[
\{ \text{pseudo-ample } M\text{-polarized K3 surfaces} \} \cong \Gamma_M \backslash \Omega_M
\]
which is an honest-to-god quasi-projective variety.

If \( M \) has a unique embedding in \( \Lambda \) upto automorphisms of \( \Lambda \), then refer to the moduli space of \( M \)-polarized K3 surfaces without fixing a specific embedding in \( \Lambda \). Note that the dimension of the moduli space is \( 20 - \text{rank}(M) \).

Easy to give a complex:description of this, but hard to give an algebraic description of this moduli space.

### 1.4. Elliptic Fibrations.

If a K3 surface \( X \) has an elliptic fibration of genus 1 from \( X \rightarrow \mathbb{P}^1 \), the fiber class \( F \) has \( F^2 = 0 \). Conversely if we have a primitive nef divisor having self-intersection zero, it gives a genus 1 fibration. Those with section correspond to embeddings of \( U \) in \( NS(X) \) and when these have reducible singular fibers, the nonidentity components give contributions to the Neron-Severi group and in particular we have the Shimura-Tate formula which tells us that
\[
\rho = 2 + \text{MW rank} + \sum (\text{No. of components} - 1)
\]
2. Orbit parametrization, generalizations of cubes involving K3 surfaces

2.1. Rubik’s revenge. We look at $4 \times 4 \times 4$ cube with entries in $k$. Taking a bijection of slices in one direction, we get a $4 \times 4$ matrix of linear forms in variables and its determinant is a quartic form in 4 variables and so it determines a quartic surface in $\mathbb{P}^3$. Generically, we get a smooth K3 surface $X$. Slices in other directions give us K3 surfaces $Y, Z$. These are isomorphic, we can think of the cube as a form on $V_1 \times V_2 \times V_3$ where $V_i$ is 4-dimensional complex vector spaces. The K3 surfaces are defined $A(x,..,z)$ being singular etc. (cf. Wei Ho’s talk) and if $x \in X$ we have $A(x,..,z)$ is singular and taking kernel in $y$ direction gives us $y$ such that $A(x,y,z) = 0$ and this implies that $y \in Y$ and we get $\phi_{XY}$ and similarly get maps $\phi_{YZ}$ and $\phi_{ZX}$, but the composition of these maps is not the identity and gives us a $K3$ surface with a genuine nontrivial automorphism.

Let’s analyze $NS(X)$. Have the hyperplane class $\mathcal{O}_{\mathbb{P}^3}(1)|_X$ and have things on $Y, Z$ which are hyperplane classes and can pull them back to $X$ and so we get three effective divisors $D_X, D_Y, D_Z$ on $X$ in $NS(X)$. If we look at the top $3 \times 3$ minor of $A(x,..,z)$ we get one coordinate of $\phi_{XY}$ as well as $\phi_{XZ}$ i.e. vanishing of this $3 \times 3$ minor should contain $D_Y, D_Z$ as well. Upshot of this turns out to be $3D_X = D_Y + D_Z$. We know that $D_X^2 = D_Y^2 = D_Z^2 = 4$ and so we deduce $D_Y \cdot D_Z = 14$. So deduce $D_X, D_Z = D_X \cdot D_Y = 6$ and so $NS(X)$ contains the lattice with form $\langle 4, 6, 6, 4 \rangle$ which has discriminant $-20$ and rank 2. Notice that the dimension of the moduli space of $4 \times 4 \times 4$ cubes is $4^3 - 3(4^3 - 1) - 1 = 18$ and so the rank of $NS(X)$ is 2.

Now we show that we can reverse this process. So $NS(X)$ is either $ZD_X + ZD_Y$ or is sublattice of index 2 (this can’t happen!), and so $D_Y$ gives you a genus 3 curve embedding into $\mathbb{P}^2$ by a linear system of degree 6 (since $D_X D_Y = 6$). To find the cube given $D_Y, D_X$ look at $H^0(X, D_X) \otimes H^0(D, D_Y) \to H^0(D_Z \otimes \mathbb{C})$. If surjective, kernel has dimension 4 if this is surjective and we get four $4 \times 4$ matrices and this gives us the $4 \times 4 \times 4$ cube.

**Theorem 2.1.** Let $V, V', V''$ be 4-dimensional spaces over some field – then the non-degenerate orbits of $\mathbb{G}_m \times SL(V) \times SL(V') \times SL(V'')$ are in bijection with isomorphism classes of $(X, L)$ where $X$ is a smooth quartic surface in $\mathbb{P}^3_k$ and $L$ is a line bundle defined over $k$ with $L^2 = 4$ and $LH = 6$ where $H$ is hyperplane of $X \subset \mathbb{P}^3$ and these are the $k$-points of moduli space of K3 surfaces of lattice polarized $\langle 4, 6, 6, 4 \rangle$.

Automorphism of going around acts on N-S by the matrix $\begin{pmatrix} -3 & -8 \\ 8 & 21 \end{pmatrix}$ and can show that it generically generates the automorphism group of a K3 surface.

2.2. $2 \times 2 \times 2 \times 4$ boxes. Let $C$ be such a one with entries in $k$. We consider the surface associated to it by the intersection of the natural six $(1, 1, 1, 1)$ forms on this tensor product. Alternatively, think of a linear combination of the four $2 \times 2 \times 2$ cubes with coefficients given by $x \in U$. We look at the locus of $x$ such that the linear combination has discriminant 0 i.e. there exists a unique linear combination in any direction such that the corresponding matrix is singular. If you project to $\mathbb{P}^3$, you get a quartic surface but has 12 singularities.
The projection to \((\mathbb{P}^1)^3\) restricts to \(X\) giving an embedding and the projection is cut out by a \(2 \times 2 \times 2\) form.

Conversely this \((2, 2, 2)\) form is zero exactly when there’s a linear combination that makes \(A(., s, t, x)\) etc vanish. What are these 12 singular points related to? Consider the projection to one of the \(\mathbb{P}^1\)’s. The generic fiber here is a \((2, 2)\) form in \(\mathbb{P}^1 \times \mathbb{P}^1 - \mathbb{A}\) general curve. Over \(r \in \mathbb{P}^1\) we have the \(2 \times 2 \times 4\) box \(A(r, ., ., .)\) which has a natural degree 4 invariant which is going to be the determinant in \(r\) considered as a \(4 \times 4\) matrix. There’s a singular point exactly when this invariant vanishes i.e. there’s a combination of these four \(2 \times 2\) matrices which is identically 0) Since we have a degree 4 polynomial in \(r\) over which the fiber is singular somewhere. INSERT PRETTY PICTURE. So we get 12 singular points, 4 for each \(\mathbb{P}^1\) and the exceptional divisors are \(-2\) curves which lie in the \(NS\) group but they do not generate it. Generically \(NS(X)\) has rank 13 and discriminant 1024.