Aspects of hadronic physics
in the gauge/gravity correspondence

Based on:

– *Hadronic Density of States from String Theory*,
  with D. Vaman.

– *Regge Trajectories Revisited in the Gauge/Gravity Correspondence*
  NPB, hep-th/0311190
  with J. Sonnenschein and D. Vaman

talk at MIT March, 2004
Motivation:

• AdS/CFT: Beyond the Supergravity Approximation. Sugra modes $\leftrightarrow$ Protected Operators.

• Sectors of Large Charge:

  BMN: Large R-charge in $\mathcal{N} = 4$ SYM $\leftrightarrow$ String shrunk to a point orbiting in $S^5$.

  GKP: Twist-two operators $\leftrightarrow$ folded string spinning in $AdS_5$.

  Tseytlin [integrable models]: Strings with two angular momenta in $S^5$ $\leftrightarrow$ $\text{Tr} Z^J \Phi^J$.

Sectors of large charges can be described by semiclassical string configurations.

Conserved classical quantities (angular momentum) $\equiv$ Quantum numbers (R-charge, spin)

Toward QCD: Look for universal quantum numbers

Properties of $\mathcal{N} = 1$ SYM that are common to YM? $U(1)_R$ is broken!

Can the density of states be computed without full knowledge of the spectrum (without fully solving string theory)? [(confining) Kutasov]

High spin states, Regge trajectories. Hadronic states in Gauge/Gravity.
Outline

• Story of density of states in Supergravity

• Annulons thermal partition function: An exact example

• Hagedorn density of states: A semiclassical approach
  (For Sugra Backgrounds dual to Confining Gauge Theories)

• Regge trajectories revisited in the Gauge/Gravity Correspondence

• The soft Pomeron trajectory: UA8 Collaboration

• Nonlinearity of Regge trajectories from string theory
Genus expansion: The intuition maker

\[ Z_{string} = e^{-2\Phi_0} Z_0 + Z_1 + e^{2\Phi_0} Z_2 + \ldots \]

\[ Z_{gauge} = N^2 Z_0 + N^0 Z_1 + \frac{1}{N^2} Z_2 + \ldots \]

• Conformal Theories: Main Contribution is \( N^2 \).

• Confining Theories: Main Contribution is \( N^0 \).
Supergravity Story Predates AdS/CFT [Klebanov]

AdS/CFT Correspondence

\[ AdS_5 \times S^5 \iff \mathcal{N} = 4 \ SU(N) \ \text{SYM} \]

The Sugra limit \( N \gg (g_{YM}^2 N)^{1/4} \gg 1 \)

\[
\begin{align*}
    ds^2 &= h^{-1/2}(r)\left[-f(r)dt^2 + dx^i dx_i\right] + h^{1/2}[f(r)^{-1}dr^2 + r^2 d\Omega_5^2]

    h(r) &= \frac{R^4}{r^4}, \quad f(r) = 1 - \frac{r_0^4}{r^4}
\end{align*}
\]

Temperature: \( T = 1/\beta = r_0^2/\pi R^2 \)

\[
S_{BH} = \frac{A_h}{4G} = \frac{\pi^2}{2} N^2 V_3 T^3 + \ldots
\]

Free \( U(N) \ \mathcal{N} = 4 \) Supermultiplet

Content: Gauge Field, \( 6N^2 \) massless scalars, \( 4N^2 \) Weyl Fermions

\[
S_0 = \frac{2\pi^2}{3} N^2 V_3 T^3.
\]

The Famous 3/4

\[
S = N^2 f(g_{YM}^2 N)VT^3
\]
A proposal for a semiclassical evaluation of $Z_1$:

The Solitonic Object for Torus Topology World Sheet

$$X^0 = n \beta \sigma_1 + m \beta \sigma_2, \quad \text{"Completion"}$$

Include quantum fluctuations around this soliton to compute $Z_1$:

$$Z \approx Z_{\text{soliton}}Z_{\text{quantum}}$$

Part I: Motivation and Justification for this proposal.
Motivating the Proposal: Compactified Boson on a Torus

- Configurations with nonzero winding number Torus \( T^2 = \mathbb{C}/\Gamma : z \sim z + \omega_1 \sim z + \omega_2 \)

\[
\Phi(z + k\omega_1 + k'\omega_2) = \Phi(z) + \beta(km + k'n), \quad k, k' \in \mathbb{Z}
\]

\((m, n)\) Specifies a Topological configuration.

\[
\Phi = \Phi_{m,n}^{cla} + \phi, \quad \Phi_{m,n}^{cla} = \beta \left( \frac{z}{\omega_1} \frac{m\bar{\tau} - n}{\bar{\tau} - \tau} - \frac{\bar{z}}{\omega_1^*} \frac{m\tau - n}{\bar{\tau} - \tau} \right) \tag{1}
\]

\(\phi\)– periodic.

\[
S[\Phi_{m,n}^{cla}] = \frac{1}{2\pi} \int dzd\bar{z} \partial\Phi_{m,n}^{cla} \bar{\partial}\Phi_{m,n}^{cla} = \beta^2 \frac{|m\tau - n|^2}{8\pi \tau_2}.
\]

Modular Invariance \(\rightarrow\) Sum over all sectors \((m, n)\)

\[
Z = \sum_{m,n} Z_{m,n} = \frac{\beta}{\sqrt{8\pi}} \frac{1}{\tau_2^{1/2} |\eta(\tau)|^2} \sum_{m,n} \exp \left( -\frac{\beta^2}{8\pi \tau_2} |m\tau - n|^2 \right).
\]

\[
Z = Z_{\text{quantum}} Z_{\text{soliton}}
\]

Q: Was the factorization an artifact of flat space?
Q: How to generalize for curved background?
Is there a solvable string theory of hadronic states?

Using a Penrose-Güven limit in Confining backgrounds.

- Generic properties of AdS Dual of confining theories
- End of Space.
- Wilson Loop shows confining behavior.

\[
T_s = \frac{1}{2\pi\alpha'} g_{tt}(r_0)
\]

- \( g_{tt}(r_0) \neq 0 \).
- \( g_{tt}(r_0) \) has a minimum (J. Sonnenschein et al.)
The Maldacena-Núñez background

- N D5 branes wrapped on $S^2$.
- IR: $\mathcal{N} = 1$ SYM contaminated with KK.

\[
ds_{str}^2 = e^{\phi_D} \left[ dx_\mu dx^\mu + \alpha' g_s N (d\rho^2 + e^{2\phi}(d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + \frac{1}{4} \sum_a (w^a - A^a)^2) \right],
\]
\[
H_{RR} = g_s N \left[ -\frac{1}{4} (w^1 - A^1) \wedge (w^2 - A^2) \wedge (w^3 - A^3) + \frac{1}{4} \sum_a F^a \wedge (w^a - A^a) \right]
\]
\[
e^{2\phi_D} = e^{2\phi_{D,0}} \frac{\sinh 2\rho}{2e\rho}, \quad e^{\phi_{D,0}} = \sqrt{g_s N}
\]
\[
e^{2\phi} = \rho \coth 2\rho - \frac{\rho^2}{\sinh^2 2\rho} - \frac{1}{4}, \quad a(\rho) = \frac{2\rho}{\sinh 2\rho}
\]
\[
A = \frac{1}{2} \left[ \sigma^1 a(\rho) d\theta_1 + \sigma^2 a(\rho) \sin \theta_1 d\phi_1 + \sigma^3 \cos \theta_1 d\phi_1 \right]
\]

$w^a$ - $SU(2)$ left-invariant one-forms

Scales associated with the $\mathcal{N} = 1$ SYM dual of the MN background.

\[
M_{gb}^2 \sim M_{KK}^2 \sim \frac{1}{g_s N \alpha'}, \quad T_s \propto M_{gb}^2 (g_s N)^{\frac{3}{2}}.
\]
The Penrose-Güven limit: Set up

Make the following change of variables

\[ dt = dx^0, \quad x^i \rightarrow \frac{1}{L} x^i, \quad \rho = \frac{m_0}{L} r, \]

\[ \theta_2 = \frac{2m_0}{L} v, \quad \phi_+ = \frac{1}{2}(\psi + \phi_2), \]

where \( L^2 = \sqrt{g_sN} \) and \( m_0 = \frac{1}{\sqrt{g_sN\alpha'}} \) is the glueball mass

\[ \hat{\phi}_1 = \phi_1 + \frac{1}{3} \phi_+ \quad \hat{\phi}_2 = \phi_2 - \phi_+. \]

\[ x^+ = t, \quad x^- = \frac{L^2}{2}(t - \frac{1}{m_0} \phi_+), \]
The Penrose-Güven Limit

$L \to \infty; m_0$ fixed

$$ds^2 = -2dx^+dx^- - m_0^2\left(\frac{1}{9}z_1^2 + \frac{1}{9}z_2^2 + v_1^2 + v_2^2\right)(dx^+)^2 + d\tilde{x}^2 + d\tilde{z}^2 + dv_1^2 + dv_2^2.$$

- 4 massless direction: (three $x$’s from WV and one $z$).
- 2 directions ($v$) with mass $m_0$.
- 2 directions with mass $\frac{1}{3}m_0$.

$$H_{RR} = -2m_0\ dx^+ \wedge \left[ dv_1 \wedge dv_2 + 1/3\ dz_1 \wedge dz_2 \right].$$

- Fermions: 4 with mass $m_0/3$ and 4 with mass $2m_0/3$

The Hamiltonian is (Poincare time/energy):

$$H = -p_+ = i\partial_+ = E - m_0\left(-\frac{1}{3}J_1 + J_2 + J_\psi\right) = E - m_0\ J,$$

$$P^+ = -\frac{1}{2}p_- = \frac{i}{2}\partial_- = \frac{m_0}{\Omega^2}\left(-\frac{1}{3}J_1 + J_2 + J_\psi\right) = m_0\frac{J}{\sqrt{g_sN}}.$$
The Annulon Hamiltonian [Gimon, LAPZ, Sonnenschein, Strassler]

The light-cone Hamiltonian of the theory has the following simple form:

\[
H = \frac{P_i^2}{2P_+} + \frac{P_4^2}{2P_+} + \frac{1}{2\alpha' P_+} \sum_{n=1}^{\infty} n(N_n^i + N_n^4)
+ \frac{1}{2\alpha' P_+} \sum_{n=0}^{\infty} \left( w_n^a(N_n^1 + N_n^2) + w_n^b(N_n^3 + N_n^4) \right)
+ \frac{1}{2\alpha' P_+} \sum_{n=0}^{\infty} \left( \omega_n^\alpha S_n^\alpha + \omega_n^\beta S_n^\beta \right). \tag{2}
\]

where \(i = 1, 2, 3, 4\), \(a = 5, 6\), \(b = 7, 8\), \(\alpha = 1, 2, 3, 4\) and \(\beta = 5, 6, 7, 8\); \(N\) and \(S\) are bosonic and fermionic occupation numbers

\[
w_n^a = \sqrt{n^2 + \left(m_0 p^+ \alpha' \right)^2}, \quad
w_n^a = \sqrt{n^2 + \frac{1}{9} (m_0 p^+ \alpha')^2},
\]

\[
\omega_n^\alpha = \sqrt{n^2 + \frac{1}{9} (m_0 p^+ \alpha')^2}, \quad
\omega_n^\beta = \sqrt{n^2 + \frac{4}{9} (m_0 p^+ \alpha')^2}. \tag{3}
\]

A string theory of hadrons

The Hamiltonian purely in Field Theory language

\[
H = \left[ \frac{P_i^2}{2m_0 J} + \frac{T_s}{2m_0 J} (N_R + N_L) \right] + \left[ \frac{T_s}{2m_0 J} (H_0 + H_R + H_L) \right].
\]
Towards the MN Annulon partition function

- building blocks: Boson off criticality [Itzykson and Saleur].

\[
\begin{align*}
Z^{(0,0)}_{lc}(\tau, m) &= \int \mathcal{D}X \exp \left[ - \int_\mathcal{T} d^2 z \bar{X} \left( -\partial_z \partial_{\bar{z}} + m^2 \right) X \right],
\end{align*}
\]

Doubly periodic quantum boson \( z = \xi_1 + \tau \xi_2 \),

\[
X(\xi_1, \xi_2) = \sum_{n_1, n_2 \in \mathbb{Z}} X_{n_1, n_2} \exp[2\pi i (n_1 \xi_1 + n_2 \xi_2)]
\]

\[
d^2z = d\xi_1 d\xi_2 \tau_2,
\]

\[
\partial_z \partial_{\bar{z}} = \frac{1}{4\tau_2^2} \left( |\tau|^2 \partial_1^2 - 2\tau_1 \partial_1 \partial_2 + \partial_2^2 \right).
\]

Explicit Gaussian integrals over \( X_{n_1, n_2} \)

\[
Z^{(0,0)}_{lc}(\tau, \mu) = \left[ \prod_{n_1, n_2 \in \mathbb{Z}} \tau_2 \left( \left( \frac{2\pi}{4\tau_2} \right)^2 |n_1 \tau - n_2|^2 + m^2 \right) \right]^{-1}.
\]

Double Product \( \longrightarrow \) Modular properties

\[
Z_{lc}(-1/\tau, m|\tau|) = Z_{lc}(\tau, m)
\]

Fermionic Partition function [antiperiodic in \( \xi_1 \)]

\[
Z^{(1/2,0)}_{lc}(\tau, \mu) = \prod_{n_1, n_2 \in \mathbb{Z}} \tau_2 \left( \left( \frac{2\pi}{4\tau_2} \right)^2 \left| 2n_1 + \frac{1}{2} \right| \bar{\tau} + n_2 \right)^2 + \frac{\mu^2 \beta^2}{\tau_2^2}
\]

Too Formal!
A Nonholomorphic Generalization of Dedekind $\eta(\tau)$

Performing one of the infinite products

$$z_{lc}^{(0,0)}(\tau,m) = \exp \left[ 2\pi \tau_2 \left( m/2 + \sum_{n=1}^{\infty} \sqrt{n^2 + m^2} \right) \right]$$

$$\left[ \prod_{n \in \mathbb{Z}} \left( 1 - \exp[2\pi i(\tau_1 n + i\tau_2 \sqrt{n^2 + m^2})] \right) \right]^{-1}.$$

Compare with

$$|\eta(\tau)|^2 = \exp \left[-\pi \tau_2 / 6\right] \left[ \prod_{n \in \mathbb{Z}} \left( 1 - \exp[2\pi i n (\tau_1 + i\tau_2)] \right) \right]^{-1}$$

(5)

$\zeta$-function regularization of the Casimir Energy

$$\gamma_0(m) = \frac{m}{2} + \sum_{n=1}^{\infty} \sqrt{n^2 + m^2} = \frac{m}{2} + \left[ -\frac{1}{12} + \frac{1}{2} m - \frac{1}{2} m^2 \ln(4\pi e^{-\gamma}) \right.$$  

$$\left. + \sum_{n=2}^{\infty} (-1)^n \frac{\Gamma(n - \frac{1}{2})}{n! \Gamma(-\frac{1}{2})} \zeta(2n - 1) m^{2n} \right],$$

$\gamma$ – Euler constant. Flat space limit ($m \to 0$)

$$\gamma_0(m) \longrightarrow \sum_{n=1}^{\infty} n = \zeta(-1) = -1/12.$$
The MN Annulon Partition Function

\[ Z(\beta, \mu) = \frac{\beta}{4\pi l_s} \int_0^\infty \frac{d\tau_2}{\tau_2^2} \int_{-1/2}^{1/2} d\tau_1 \sum_{r=1}^{\infty} [1 - (-1)^r] \exp\left( -\frac{\beta^2 r^2}{2\pi \alpha' \tau_2} \right) \]
\[ \times \left[ \tau_2^{-1/2} |\eta(\tau)|^{-2} \right]^4 \quad 4 \text{ massless bosons} \]
\[ \times \left[ z_{lc}^{(0,0)}(\tau, \frac{m_0 \beta r}{\tau_2}) \right]^2 \quad 2 \text{ } m_0 \text{ bosons} \]
\[ \times \left[ z_{lc}^{(0,0)}(\tau, \frac{m_0/3 \beta r}{\tau_2}) \right]^2 \quad 2 \text{ } m_0/3 \text{ bosons} \]
\[ \times \left[ z_{lc}^{(1/2,0)}(\tau, \frac{m_0/3 \beta r}{\tau_2}) \right]^4 \quad 4 \text{ } m_0/3 \text{ fermions} \]
\[ \times \left[ z_{lc}^{(1/2,0)}(\tau, \frac{2m_0/3 \beta r}{\tau_2}) \right]^4 \quad 4 \text{ } 2m_0/3 \text{ fermions} \]
\[ + \text{ stuff associated with a nonsupersymmetric ground state} \quad (6) \]
Hagedorn Temperature:
\[-\frac{T_s \beta_H^2}{2\pi} + \frac{2}{3} - 4\pi \gamma_0(m_0 \beta_H) - 4\pi \gamma_0 \left(\frac{m_0}{3} \beta_H\right) + 8\pi \gamma_{1/2} \left(\frac{m_0}{3} \beta_H\right) + 8\pi \gamma_{1/2} \left(\frac{2m_0}{3} \beta_H\right) = 0. \tag{7}\]

Limits: \( m_0 \to 0 \) [IIB Strings in Flat Space]

Large \( m_0 \) a lower dimensional theory with \( \beta_H = \frac{2\pi}{(\sqrt{3}T_s^{1/2})} \)

Density of States for the MN Annulons

\[ S = \frac{2\pi}{\sqrt{3} \tilde{T}_s^{1/2}} \cdot \]

\[ \tilde{T}_s = T_s / J. \] In general \( S(m_0, T_s / J) \).
Where are the Temporal Windings?

• Using Light-Cone, temporal coordinate gauged away, RR field.
• How can this result be turned into evidence for the proposal.

The above Partition function was written over the strip:

\[ E : \quad \tau_2 > 0, \quad -\frac{1}{2} < \tau_1 < \frac{1}{2}. \]

Generalizing a flat space result: Tiling.

\[
Z(\beta, \mu) = \frac{\beta}{4\pi l_s} \int \frac{d\tau_2}{\tau_2^2} \int d\tau_1 \sum_{m,n} \prod_{n_1,n_2 \in \mathbb{Z}} \exp \left( -\frac{\beta^2 |m\tau + n|^2}{2\pi \alpha' \tau_2} \right) \\
\times \left[ \tau_2^{-1/2} |\eta(\tau)|^{-2} \right]^4
\]

\[
\left[ \frac{\pi^2}{4\tau_2} |n_1\tau - n_2|^2 + \frac{\mu^2 \beta^2}{\tau_2} |m\tau + n|^2 \right]^{-2}
\]

\[
\left[ \frac{\pi^2}{4\tau_2} |n_1\tau - n_2|^2 + \frac{\mu^2 \beta^2}{9\tau_2} |m\tau + n|^2 \right]^{-2}
\]

\[
\left[ \frac{\pi^2}{4\tau_2} \left| \frac{2n_1 + 1}{2} \tau + n_2 \right| + \frac{\mu^2 \beta^2}{9\tau_2} |m\tau + n|^2 \right]^4
\]

\[
\left[ \frac{\pi^2}{4\tau_2} \left| \frac{2n_1 + 1}{2} \tau + n_2 \right| + \frac{4\mu^2 \beta^2}{9\tau_2} |m\tau + n|^2 \right]^4
\]

\[+ \text{ stuff associated with a nonsupersymmetric ground state} \quad (8)\]

\[m, n \in \mathbb{Z}^*\]
Integration over the fundamental domain:

\[ \mathcal{F} : \quad |\tau| > 1, \quad -\frac{1}{2} < \tau_1 < \frac{1}{2}. \]

Mixing \( Z_{\text{quantum}} Z_{\text{soliton}} \): \((m,n)\) dependent masses:

\[ \mu \rightarrow \mu |m\tau + n| \]
Temporal winding modes as solitons

The World Sheet has Torus Topology:

\[ ds^2 = |d\sigma_1 + \tau d\sigma_2|^2 = d\sigma_1^2 + |\tau|^2 d\sigma_2^2 + 2(\text{Re}\ \tau) d\sigma_1 d\sigma_2. \]

String Action [bosonic]:

\[
I = \frac{1}{2\pi\alpha'} \int d\sigma_1 d\sigma_2 \sqrt{\gamma} g_{\mu\nu} \gamma^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu.
\]

EOM [assumption: \( g_{\mu\nu}(r) \)]:

\[
\partial_\alpha (\sqrt{\gamma} \gamma^{\alpha\beta} g_{00} \partial_\beta X^0) = 0.
\]

\[
\partial_\alpha (\sqrt{\gamma} \gamma^{\alpha\beta} g_{rr} \partial_\beta r) - \frac{1}{2} \partial_r g_{00} [\sqrt{\gamma} \gamma^{\alpha\beta} \partial_\alpha X^0 \partial_\beta X^0] = 0.
\]
Looking for the winding Soliton:

\[ X^0 = m\beta\sigma_1 + n\beta\sigma_2, \quad r = r(\sigma_1, \sigma_2). \]

Complicated in General

\[ \exists r_0 : \quad g_{00}(r_0) \neq 0, \quad \partial_r g_{00}(r_0) = 0. \]

• For confining backgrounds \( r = r_0 \) is a solution [String at the AdS Wall] \( T_s = g_{00}(r_0)/2\pi\alpha' \)

\[ S_\beta(m, n) = T_s \frac{1}{\tau_2} \left[ \beta^2 (n^2 + m^2|\tau|^2 - 2(\text{Re}\, \tau) mn) \right] = T_s \frac{\beta^2 |m\tau - n|^2}{\tau_2}. \]
Sketch of Fluctuations:

• $r_0$ is now $\tau = 0$.
• $e_3^2 + e_4^2 + e_5^2|_{\tau = 0} = \frac{1}{2} d\Omega_3^2$ round $S^3$ with radius $1/\sqrt{2}$.
• $S^3(\theta, \phi, \psi)$ by fixing $\theta = \pi/2$ becomes $\mathbb{R}^3(y^1, y^2, y^3)$.
• $e^2g|_{\tau = 0} \approx \tau^2 \rightarrow \tau$-direction with $S^2(\theta_1, \phi_1)$ into $\mathbb{R}^3(\tau^1, \tau^2, \tau^3)$

\[ S_{2b} = S[X^0_{classical}, r = r_0] + \frac{1}{2\pi \alpha'} \int d\sigma_1 d\sigma_2 \sqrt{\gamma} \gamma^{\alpha \beta} (\partial_\alpha X^a \partial_\beta X^a g_{00} + \alpha' g_s N g_{00} [\partial_\alpha \tau^i \partial_\beta \tau_i + \frac{1}{4} \partial_\alpha y^i \partial_\beta y_i] + \frac{4\beta^2}{9 |\text{Im}\tau|^2} g_{00} |m\tau - n|^2 \tau^i \tau_i) \] (9)

where $a = 1, \ldots, 4$ and $i = 1, 2, 3$.

• The mass $(2/3)\beta \sqrt{\frac{1}{\alpha' g_s N}} |m\tau - n|/\text{Im}\tau$.

\[ S_{2f} = \frac{i}{2\pi \alpha'} \int \bar{\theta}^I (\sqrt{\gamma} \gamma^{\alpha \beta} \delta^{IJ} - \epsilon^{\alpha \beta} \sigma_3^{IJ}) \partial_\alpha X^0 \Gamma_0 e_0^0 (\delta^{JK} \partial_\beta + \frac{1}{8 \cdot 3!} e^\phi \sigma_1^{JK} \Gamma_{\mu_1 \mu_2 \mu_3} F_{\mu_1 \mu_2 \mu_3} \partial_\beta X^0 \Gamma_0 e_0^0) \theta^K \] (10)

\[ F(3) = -\frac{1}{4} g_s N dy_1 \wedge dy_2 \wedge dy_3 \]

Choose the $\kappa$ gauge: $\theta^1 = \theta^2$
Hagedorn Behavior

- Partition function:

\[
Z_{T^2} = \sum_{m,n \in \mathbb{Z}}' \frac{e^{-\beta g_{00}} |m\tau - n|^2}{4\pi\alpha'} \int\mathcal{F} d^2\tau e^{-\beta g_{00} |m\tau - n|^2} z_{0,0}(\tau,0)^5 z_{0,0}(\tau, M^2 = \frac{4\beta^2 |m\tau - n|^2}{Im\tau^2} \frac{1}{\alpha' g_s N})^3 z_{b_1,b_2}(\tau,0)^8 \tag{11}
\]

\[
Z_{T^2} \approx \int e^{-\beta g_{00} |m\tau|^2} e^{-\pi Im\tau \sum_{l \in \mathbb{Z}} (5l + 3\sqrt{l^2 + 4\beta^2} \frac{1}{\alpha' g_s N} - 8(l+1/2))}, \tag{12}
\]

\[
T_H:\]

\[
\frac{1}{4\pi\alpha'} \beta_H g_{00} = -2\pi \big(5\gamma_0(0) + 3\gamma_0(2\beta_H \sqrt{\frac{1}{\alpha' g_s N}/3}) - 8\gamma_{1/2}(0)\big). \tag{13}
\]

\[
d(E) \approx \exp \left(\sqrt{3\pi} \frac{E}{T_s^{1/2}}\right). \tag{14}
\]

The Density of states depends on the gauge theory quark-antiquark string tension
Regge Trajectories Revisited in the Gauge/Gravity Correspondence

- A Regge trajectory: a line in the Chew-Frautschi plot: \( J = \alpha_0 + \alpha' t \)

- Well described by simple strings model but now we have the right string models.

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<th>String Theory Configuration</th>
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<tr>
<td>Baryons of heavy quarks</td>
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<tr>
<td>Mesons of light quarks</td>
<td>Spinning open strings ending on D7</td>
</tr>
</tbody>
</table>

States in gauge theory and their corresponding classical configuration in the string theory
Closed spinning strings in supergravity backgrounds

Regge trajectories for Glueballs
Closed spinning strings in confining theories

\[ ds^2 = h(r)^{-1/2} \left[ -dt^2 + dx_1^2 + dx_2^2 + dx_3^2 \right] + h(r)^{1/2}dr^2 + \ldots \]  \hfill (15)

The relevant classical equations of motion are

\[ \partial_a (h^{-1/2} \eta^{ab} \partial_b t) = 0, \]
\[ \partial_a (h^{-1/2} \eta^{ab} \partial_b x^i) = 0, \]
\[ \partial_a (h^{1/2} \eta^{ab} \partial_b r) = \frac{1}{2} \partial_r (h^{-1/2}) \eta^{ab} \left[ - \partial_a t \partial_b t + \partial_a x_i \partial_b x^i \right]. \]  \hfill (16)

- Conditions for confinement in gauge/gravity: \( g_{00} \) has a nonzero minimum at some point \( r_0 \).
\[ \partial_r (g_{00})|_{r=r_0} = 0, \quad g_{00}|_{r=r_0} \neq 0. \]  \hfill (17)

\[ t = e^\tau, \quad x_1 = \frac{1}{\omega} \cos e\omega \tau \sin e\omega \sigma \quad x_2 = \frac{1}{\omega} \sin e\omega \tau \sin e\omega \sigma \]  \hfill (18)

\[ E = \frac{g_{00}(r_0)}{2\pi\alpha'} \frac{1}{\omega} \frac{\pi}{2}, \quad J = \frac{g_{00}(r_0)}{2\pi\alpha'} \frac{1}{\omega^2} \frac{\pi}{4}. \]  \hfill (19)

Typical Regge trajectories

\[ E^2 = 4\pi T_s S, \quad \text{or} \quad J = \frac{\alpha'}{2} t \]  \hfill (20)

Problems

- This trajectory has zero intercept.
- It is strictly linear.
The soft Pomeron: UA8 Collaboration

- Experimental suggestion
  \[ \alpha(t) = 1.10 + 0.25 t + \alpha'' t^2, \]
  \[ \alpha'' = 0.079 \pm 0.012 \text{GeV}^{-4} \]

- Positive Nonvanishing intercept.
- Positive curvature \( \alpha'' > 0 \).

The slope from sugra data

<table>
<thead>
<tr>
<th>State</th>
<th>((\text{Mass})^2/\varepsilon^{4/3})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0++</td>
<td>9.78</td>
</tr>
<tr>
<td>0+*</td>
<td>33.17</td>
</tr>
<tr>
<td>1--</td>
<td>14.05</td>
</tr>
<tr>
<td>1--*</td>
<td>42.90</td>
</tr>
<tr>
<td>2++</td>
<td>18.33</td>
</tr>
</tbody>
</table>

- This table was obtained in 2000 [Cácers and Hernández], state in red 04 [Amador and Cáceres].
- The prediction for the Regge slope
  \[ J = \alpha(t) = 0.234 t + \alpha_0 \]
Semiclassical quantization

- Compute how the classical energy changes (similar to Lüschterm):

\[ e \Delta E = \int d\sigma < \Psi | \mathcal{H}(\delta X) | \Psi > = \text{sum of zero-point energies} \]

- New feature of confining backgrounds? [Compared to strings in flat space]

\[ \gamma^{\tau\tau} g_{tt} \partial_\tau t \partial_\tau t + \gamma^{\alpha\beta} \partial_\alpha x^i \partial_\beta x^i g_{ii} = \left( \frac{8 e^{\phi_0}}{9} \kappa^2 \cos^2 \omega \sigma \right) \tau^i \tau_i \]

\[ [\partial_\tau^2 - \partial_\sigma^2 + m_0^2 \cos^2(\omega \sigma)] \delta \tau_i = 0. \]

Mathieu differential equation

\[ \lambda_{r,n} = \frac{n^2}{\omega^2} + \frac{m_0^2}{2 \omega^2} + \frac{r^2}{\omega^2} + \frac{1}{2(r^2 - 1)} \frac{m_0^4}{16 \omega^4} + \mathcal{O}(m_0^8), \]

- Contribution to the zero point energy

\[ \Delta E = -\frac{1}{12} + m_0. \]

- Fermions

\[ S_F \approx \frac{i}{2} T_s \int \sqrt{\gamma} \gamma^{\alpha\beta} \left( \bar{\theta} \partial_\alpha \tilde{X}^\mu \Gamma_\mu \partial_\beta \theta + \frac{1}{4} \partial_\alpha \tilde{X}^\mu \partial_\beta \tilde{X}_\nu \theta \gamma^\mu_\beta \hat{f} \gamma^\nu_\mu \right), \quad \hat{f}^2 = 2 e^2 \ell^2 \cos^2 \omega \sigma \]
Nonlinear Regge Trajectories

\[ E - E_{\text{Class}} = \pi \left( \frac{3}{2} m_0 - 4\ell \right) = z_0 \]

<table>
<thead>
<tr>
<th>( m_0 )</th>
<th>Klebanov-Strassler</th>
<th>Maldacena-Núñez</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{3^{1/6} a_1^{1/2}}{a_0} )</td>
<td>( \frac{\varepsilon^{2/3}}{g_s M\alpha'} )</td>
<td>( \frac{2}{3} \sqrt{g_s N\alpha'} )</td>
</tr>
<tr>
<td>( \frac{3^{1/2}}{2^{1/6} a_0} )</td>
<td>( \frac{g_s^{-1} \varepsilon^{2/3}}{g_s M\alpha'} )</td>
<td>( \frac{2^{1/2}}{g_s N \sqrt{g_s N\alpha'}} )</td>
</tr>
</tbody>
</table>

\[ J = \frac{1}{2} \alpha' E^2 - \alpha' z_0 E + \frac{1}{2} \alpha' z_0^2. \]

\[ J \equiv \alpha(t) = \alpha_0 + \frac{1}{2} \alpha' t + \beta \sqrt{t} \]

- Positive Nonvanishing intercept.
- Positive curvature \( \alpha''(t) > 0 \).
Outlook:

- Exact Calculation of the Density of States in Hadronic String Theories.

- A proposal for how to compute the Hagedorn Density of States when the full string solution is not available.

- How about transitions: Confinement/Deconfinement?

- Nonzero intercept and nonlinearity of glueball trajectories.

- Regge trajectory for dynamical mesons (light masses). Description of the finer structure.

- What other hadronic properties can one get a handle on?