1. Consider a particle in the infinite square well potential

\[ V(x) = \begin{cases} 
0, & \text{if } 0 \leq x \leq a \\
\infty, & \text{otherwise}
\end{cases} \]  

(0.1)

Show that the possible values of the energy are given by

\[ E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2}. \]  

(0.2)

**Instruction** Show your work, make sure you include the Schrödinger equation, its solutions satisfying the natural boundary conditions and how they lead to quantization.

2. In deriving the Generalized Uncertainty Principle we consider two observables \( A \) and \( B \) and their corresponding uncertainties given by

\[ \sigma_A^2 = \langle (\hat{A} - \langle A \rangle)^2 \rangle \]  

(0.3)

\[ \sigma_B^2 = \langle (\hat{B} - \langle B \rangle)^2 \rangle \]  

(0.4)

where

\[ |f\rangle \equiv (\hat{A} - \langle A \rangle)|\Psi\rangle, \quad \text{and} \quad |g\rangle \equiv (\hat{B} - \langle B \rangle)|\Psi\rangle. \]  

(0.5)

The necessary and sufficient condition for achieving minimum of uncertainty (minimum wave packet) is

\[ |g\rangle = ia|f\rangle, \quad a \text{ real}. \]  

(0.6)

Show that for position-momentum uncertainty the criterion (0.6) implies that the minimum-uncertainty wave packet is a Gaussian, i.e., the wave function that minimizes the uncertainty principle is a Gaussian.

3. Show that if \( f \) is simultaneously an eigenfunction of \( L^2 \) and of \( L_z \), that is,

\[ L^2 f = \lambda f, \quad \text{and} \quad L_z f = \mu f, \]  

(0.7)

then the square of the eigenvalue of \( L_z \) cannot exceed the eigenvalue of \( L^2 \).
4. An electron is in the spin state
\[ \chi = A \begin{pmatrix} 3i \\ 4 \end{pmatrix} \]. \hspace{1cm} (0.8)

i) Determine the normalization constant \( A \).

ii) Find the expectation value of \( S_x \) and \( S_z \).

5. Suppose you had three particles in a one-dimensional harmonic oscillator potential, in thermal equilibrium, with total energy \( E = (9/2) \hbar \omega \). If they are distinguishable particles (but all with the same mass), (i) what are the possible occupation-number configurations? (ii) What is the most probable configuration?

6. In the two-dimensional infinite potential well the energies are given by
\[ E_{n_x n_y} = \frac{\pi^2 \hbar^2}{2m} \left( \frac{n_x^2}{l_x^2} + \frac{n_y^2}{l_y^2} \right) \]. \hspace{1cm} (0.9)
Assuming that the number of free electrons per unit area is \( \sigma \), calculate the Fermi energy for electrons in a two-dimensional infinite square well.

7. The most prominent feature of the hydrogen spectrum in the visible region is the red Balmer line, coming from the transition \( n = 3 \) to \( n = 2 \). i) Determine the wavelength and the frequency of this line. The fine structure, that is, the inclusion of the spin-orbit coupling and the consideration of the relativistic correction yield a correction to the Bohr energies of the form
\[ E_{fs} = \frac{E_n^2}{2mc^2} \left( 3 - \frac{4n}{j + 1/2} \right) \]. \hspace{1cm} (0.10)
ii) Determine how many sublevels the \( n = 3 \) level splits into and find \( E_{fs} \) for each of them.

8. The correction to the energy due to the hyperfine splitting in deuterium is
\[ E_{hf} = \frac{\mu_0 g_e e^2}{3\pi m_d m_e a^3} < S_d \cdot S_e > \]. \hspace{1cm} (0.11)
Calculate the wavelength of the photon emitted under the hyperfine transition in the ground state (\( n = 1 \)) of deuterium. Note Deuterium is hydrogen with an extra neutron. The proton and neutron bind together with spin 1 and \( g \)-factor 1.71.