Superficial Divergences

Let us consider \( \varphi^3 \) scalar field theory in \( d = 4 \) dimensions. The Lagrangian for this theory is
\[
\mathcal{L} = \frac{1}{2} (\partial_\mu \varphi)^2 - \frac{1}{2} m^2 \varphi^2 - \frac{1}{3!} \lambda \varphi^3.
\]

a) Let us determine the superficial divergence \( D \) for this theory in terms of the number of vertices \( V \) and the number of external lines \( N \). From this we are to show that the theory is super-renormalizable.

In generality, the superficial divergence of a \( \varphi^n \) theory in \( d \) dimensions can be given by
\[
D = dL - 2P,
\]
where \( L \) is the number of loops and \( P \) is the number of propagators because each loop contributes a \( d \)-dimensional integration and each propagator contributes a power of 2 in the denominator. Furthermore, we see that \( nV = N + 2P \) because each external line connects to one vertex and each propagator connects two and each vertex involves \( n \) lines. This implies that \( P = \frac{1}{2} (nV - N) \).

Therefore, still in complete generality, the superficial divergence of a \( \varphi^n \) theory in \( d \)-dimensions may be written
\[
D = dL - 2P = d \frac{d}{2} nV - \frac{d}{2} N - dV + d - nV + N,
\]
\[
= d + \left( n \frac{d}{2} - 2 \right) V - \frac{d}{2} N.
\]

Therefore, in a 4-dimensional \( \varphi^3 \)-theory the superficial divergence is given by
\[
D = 4 - V - N.
\]

We see that because \( D \propto -V \) the theory is super-renormalizable.

b) We are to show the superficially divergent diagrams for this theory that are associated with the exact two-point function.

Using equation (1.a) above, we see that the three superficially divergent diagrams in this \( \varphi^3 \)-theory associated with the exact two-point function are:

\[ \includegraphics[width=0.2\textwidth]{superficial_diagram.png} \]

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c) Let us compute the mass dimension of the coupling constant \( g \).

Because \( \mathcal{L} \) must have dimension (mass)\(^4\), each term should have dimension (mass)\(^4\).

Because of the \( m^2 \varphi^2 \) term, this implies that the field \( \varphi \) has dimension (mass)\(^1\).

Therefore the coupling \( g \) must have dimension (mass)\(^1\).
In the limit where $q \to 0$, we see that this implies

$$\pi(p)\delta \Gamma(u(p)) = \frac{\lambda^2}{2} \int \frac{d^d k}{(2\pi)^d} \frac{\pi(p) \gamma^\mu (k^\mu + m)}{(p - k)^2 - m_0^2 + i\epsilon}(k^2 - m^2 + i\epsilon).$$

Using Feynman parametrization to simplify the denominator, we will use the variables

$$\ell \equiv k - z p \quad \text{and} \quad \Delta \equiv (1 - z)^2 m^2 + zm_0^2.$$

The numerator of the integrand is then reduced to

$$\mathcal{N} = \pi(p) \gamma^\mu (k^\mu + m) \gamma^\nu (k^\nu + m) \mu(u(p)),$$

$$= \pi(p) \left[ \gamma^\mu (k^\mu + z^2 \phi^\mu + m z^2 \phi^\mu + m z \gamma^\mu \phi^\nu + m^2 \gamma^\mu \phi^\nu + m^2 \gamma^\mu \gamma^\nu \phi^{\lambda} \right] \mu(u(p)),$$

$$= \pi(p) \left[ \frac{d}{d} (2 \gamma^\mu - d \gamma^\nu) + z^2 m^2 \gamma^\mu + m^2 z \gamma^\mu + m^2 \gamma^\nu \right] u(p),$$

$$= \pi(p) \left[ \gamma^\mu \left( \frac{2 - d}{d} \ell^2 + m^2 (1 + z^2) \right) \right] u(p).$$

Combining this with our work above, we see that this implies

$$\delta Z_1 = -\delta F_1(q = 0) = -i \lambda^2 \left( 1 - \frac{2 - d}{d} \frac{2}{\ell^2 - \Delta + i\epsilon} \right) \mu(u(p)),$$

$$= \frac{\lambda^2}{2} \int_0^1 dz (1 - z^2) \left[ 2 - d \frac{2}{\ell^2 - \Delta + i\epsilon} \right] \mu(u(p)).$$

Let us now compute the one-loop contribution of $\phi$ to the electron two-point function,

$$e^- \xrightarrow{p - k} p \xrightarrow{\ell} k \xrightarrow{\Delta} p \xrightarrow{\mu = m} \mu_2$$

We will define the following variables for Feynman parametrization of the denominator:

$$\ell \equiv k - z p, \quad \text{and} \quad \Delta \equiv -z^2 m^2 + zm_0^2 + (1 - z) m^2.$$

We see therefore that

$$\Sigma_{\phi_2} = \frac{\lambda^2}{2} \int \frac{d^d k}{(2\pi)^d} \left[ (p - k)^2 - m_0^2 + i\epsilon \right] \frac{\left( \frac{2}{\ell^2 + \Delta} \right)}{(2\gamma^\mu \phi^\nu + m z \gamma^\mu \phi^\nu + m^2 \gamma^\mu \phi^\nu + m^2 \gamma^\mu \gamma^\nu \phi^{\lambda} \right)}.$$

Therefore,

$$\delta Z_2 = \frac{\partial \Sigma_{\phi_2}}{\partial \mu} \bigg|_{\mu = m} = -\frac{\lambda^2}{2} \int_0^1 dz \left[ \frac{2}{\ell^2 - \Delta - \gamma E + \log(4\pi)} + \frac{2m z (1 - z)}{\Delta} \right].$$

$$\therefore \delta Z_2 = -\frac{\lambda^2}{2} \int_0^1 dz \left[ \frac{2}{\ell^2 - \Delta - \gamma E + \log(4\pi)} + \frac{2m z (1 - z)(1 - z)}{\Delta} \right].$$

(2.a.2)
$\delta Z_2 - \delta Z_1 = \frac{\lambda^2}{32\pi^2} \int_0^1 dz \left[ \left(1 - 2z\right) \log \left(\frac{1}{\Delta}\right) + (1 - 2z) \left(\frac{2}{e} - \gamma_E + \log(4\pi)\right) - (1 - z) - \frac{m^2(1 - z)(1 + z)}{\Delta} (2z - (1 + z)) \right],$

$= \frac{\lambda^2}{32\pi^2} \int_0^1 dz \left[ (1 - 2z) \log \left(\frac{1}{\Delta}\right) - (1 - z) + \frac{m^2(1 - z)^2(1 + z)}{\Delta} \right].$

$= \frac{\lambda^2}{32\pi^2} \int_0^1 dz \left[ (1 - z) - \frac{m^2(1-z)(1-z^2)}{\Delta} - (1 - z) + \frac{m^2(1-z)^2(1+z)}{\Delta} \right].$

$= \frac{\lambda^2}{32\pi^2} \int_0^1 dz \left[ -\frac{m^2(1-z)^2(1+z)}{\Delta} + \frac{m^2(1-z^2)(1+z)}{\Delta} \right].$

We can expect that $Z_1 = Z_2$ quite generally in this theory because our proof of the Ward-Takahashi identity relied, fundamentally, on the local $U(1)$ gauge invariance of the $A_\mu$ term in the Lagrangian which is not altered by the addition of the scalar $\phi$.

b) Let us now consider the renormalization of the $\bar{\psi}\phi\psi$ vertex in this theory.

The two diagrams at the one-loop level that contribute to $\pi(p')\delta\Gamma u(p)$ are

\begin{center}
\begin{tikzpicture}

\draw[thick] (0,0) -- (1,1) node[midway,above]{$p'$} node[below]{$p - k$} ;
\draw[thick] (1,0) -- (2,1) node[midway,above]{$p'$} node[below]{$p - k$} ;
\draw[thick] (1,1) -- (2,1) node[midway,above]{$k' = k + q$} ;
\draw[thick] (1,0) -- (2,0) node[midway,above]{$k = k + q$} ;
\draw[thick] (2,1) -- (3,1) node[midway,above]{$p - k$} node[below]{$k$} ;
\draw[thick] (2,0) -- (3,0) node[midway,above]{$p - k$} node[below]{$k$} ;
\draw[thick] (3,1) -- (4,1) node[midway,above]{$q$} ;
\draw[thick] (3,0) -- (4,0) node[midway,above]{$q$} ;
\draw[thick] (4,1) -- (5,1) node[midway,above]{$p'$} node[below]{$p$} ;
\draw[thick] (4,0) -- (5,0) node[midway,above]{$p$} node[below]{$p$} ;
\end{tikzpicture}
\end{center}

These diagrams yield

$$\pi(p')\delta\Gamma u(p) = \int \frac{d^4k}{(2\pi)^4} \pi(p') \left[ \left(-i\frac{\lambda}{\sqrt{2}}\right) \frac{i}{(p - k)^2 - m^2 + i\epsilon} \frac{i(k + q + m)}{(k + q)^2 - m^2 + i\epsilon} \frac{i(k + m)}{(k^2 - m^2 + i\epsilon)} \left(-i\frac{\lambda}{\sqrt{2}}\right) \right] u(p).$$

Taking the limit where $q \to 0$ and introducing the variables

$$\ell \equiv k - z\rho, \quad \Delta_1 \equiv (1 - z)^2m^2 + z^2m^2_0, \quad \text{and} \quad \Delta_2 \equiv (1 - z)^2m^2 + z^2\mu^2,$$

this becomes,

$$\pi(p')\delta\Gamma u(p) = \int_0^1 dz(1 - z) \int \frac{d^4\ell}{(2\pi)^4} \pi(p) \left[ i\lambda^2 \ell^2 + (1 + z)^2m^2 \left(\frac{\Gamma(2 - \Delta_1 + i\epsilon)^2}{(\ell^2 - \Delta_1 + i\epsilon)^3} - 2i\epsilon^2 \frac{d\ell^2 + m^2 (d z^2 + 1) + 2z(2 - d)}{(\ell^2 - \Delta_2 + i\epsilon)^3} \right) u(p).$$

Therefore,

$$\delta Z_1' = -\delta F_1' = \int_0^1 dz(1 - z) \left[ -i\lambda^2 \ell^2 + (1 + z)^2m^2 \left(\frac{\Gamma(2 - \Delta_1 + i\epsilon)^2}{(\ell^2 - \Delta_1 + i\epsilon)^3} + 2i\epsilon^2 \frac{d\ell^2 + m^2 (d z^2 + 1) + 2z(2 - d)}{(\ell^2 - \Delta_2 + i\epsilon)^3} \right) \right],$$

$$= \int_0^1 dz(1 - z) \left[ \frac{\lambda^2}{16\pi^2} \frac{d}{d\epsilon} \Gamma\left(2 - \frac{\epsilon}{2}\right) - \frac{\epsilon}{2} \frac{d}{d\epsilon} \Gamma\left(2 - \frac{\epsilon}{2}\right) \right] + \text{finite terms},$$

$$= \int_0^1 dz(1 - z) \left[ \frac{\lambda^2}{16\pi^2} \frac{2}{\epsilon} - \log \Delta_1 - \gamma_E + \log(4\pi) - \frac{1}{2} \right] + \frac{2\alpha}{\pi} \left(\frac{2}{\epsilon} - \log \Delta_2 - \gamma_E + \log(4\pi) - 1\right) + \text{finite terms},$$

$$= \int_0^1 dz(1 - z) \frac{\lambda^2}{16\pi^2} - \frac{2\alpha}{\pi} + \text{finite terms.} \quad (2.b.2)$$
Now let us compute $\delta Z_2'$. We see that this factor comes from the diagrams,

\[
\begin{array}{c}
\text{\large $e^-$} \\
p & k & p
\end{array}
\quad + \quad
\begin{array}{c}
\text{\large $e^-$} \\
p & k & p
\end{array}
\]

We see that we have already computed both of these contributions; the first diagram’s contribution was computed above and the second diagram’s contribution was computed in homework 6.

Therefore, we note that

\[
\delta Z_2' = \frac{1}{\epsilon} \left( -\frac{\lambda^2}{32\pi^2} - \frac{\alpha^2}{2\pi} \right) + \text{finite terms.}
\]  

(2.b.3)

Combining these results, we have that

\[
\therefore \delta Z_2' - \delta Z_1' = \frac{3}{\epsilon} \left( \frac{\alpha}{2\pi} - \frac{\lambda^2}{32\pi^2} \right) + \text{finite terms} \neq 0.
\]  

(2.b.4)

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