The Decay of Vector into Two Scalars

We are to compute the decay rate of unpolarized vector particles of mass $M$ into two scalars of mass $m$. We should calculate the decay rate in the rest frame.

Defining $\tilde{p}^\mu = (\bar{p} - p)^\mu$, the amplitude for the decay diagram is given by

$$iM = \epsilon_\mu i f \tilde{p}^\mu.$$ 

It is quite straightforward to calculate the spin-averaged square of the amplitude,

$$|M|^2 = \frac{1}{3} \sum_{\text{spin}} \epsilon_\mu i f \tilde{p}^\mu \epsilon_\nu^* (-i) f \tilde{p}^\nu,$$

$$= \frac{f^2}{3} \left( \frac{k_\mu k_\nu}{M^2} - g_{\mu\nu} \right) \tilde{p}^\mu \tilde{p}^\nu,$$

$$= \frac{f^2}{3} \left( \frac{(k_\mu \tilde{p}^\mu)^2}{M^2} - \tilde{p}^2 \right).$$

Now, because we are computing this in the rest frame where $k_\mu = (M, 0)$ and $\tilde{p}^\mu = (0, -2|\vec{p}|)$, $k_\mu \tilde{p}^\mu = 0$.

Similarly, we know that $\tilde{p}^2 = 4|\vec{p}|^2$. Therefore,

$$|M|^2 = \frac{4f^2|\vec{p}|^2}{3}.$$

Note that $|\vec{p}| = E^2 - m^2 = \left( \frac{M^2}{4} - m^2 \right)^{1/2}$. Using this and the equation for the decay rate found in Peskin and Schroeder,

$$\Gamma = \frac{1}{2M} \int \frac{d\Omega}{16\pi^2} |\vec{p}| \frac{|M|^2}{M},$$

$$= \frac{f^2}{24\pi^2 M^2} \int d\Omega |\vec{p}|^3,$$

$$\therefore \Gamma = \frac{f^2 \left( \frac{M^2}{4} - m^2 \right)^{3/2}}{6\pi M^2}.$$

Mott’s Formula

We are to generalize problem 2 of Homework 8 in the relativistic case. We computed then the general amplitude to be

$$\mathcal{M} = \frac{-ie^2 Z}{(p_f - p)^2} \bar{u}^s(p_f) \gamma^0 u^s(p).$$

To compute the spin averaged amplitude, it will be helpful to recall our earlier kinematic result that $(p_f - p)^4 = 16|\vec{p}|^4 \sin^4 \theta/2$. Let us now compute the amplitude squared in the spin-averaged case.

$$|\mathcal{M}|^2 = \frac{1}{2} \frac{Z^2 e^4}{(p_f - p)^2} \sum_{\text{spin}} \bar{u}^s(p) \gamma^0 u^s(p_f) \bar{u}^s(p_f) \gamma^0 u^s(p),$$

$$= \frac{Z^2 e^4}{32|\vec{p}|^4 \sin^4 \theta/2} \text{Tr} \left( \gamma^0 p_f + m \right) \gamma^0 \phi + m \right).$$

It will be helpful to break up the trace into its four additive pieces.

$$\text{Tr} \left( \gamma^0 p_f + m \right) \gamma^0 \phi + m \right) = \text{Tr} \left( \gamma^0 p_f \gamma^0 \phi \right) + \text{Tr} \left( \gamma^0 m \gamma^0 \phi \right) + \text{Tr} \left( \gamma^0 p_f \gamma^0 m \right) + \text{Tr} \left( \gamma^0 m \gamma^0 m \right).$$
It should be clear that the two middle terms are both zero because there is an odd number of \( \gamma \)'s. The last term is nearly trivial, \( \text{Tr} (\gamma^0 m \gamma^0 m) = 4m^2 \). Let us now work on the first term.

\[
\text{Tr} (\gamma^0 \gamma^\nu \gamma^0 \gamma^\nu) = p_{f\mu} p_{\nu} \text{Tr} (g^{\mu \nu} g^{00} g^{\rho \sigma} + g^{00} g^{\rho \sigma} + g^{00} g^{00}),
\]

\[
= 4 (2E^2 - p_{f\mu} p^{\mu}),
\]

\[
= 4 (2E^2 - E^2 + \vec{p} \cdot \vec{p}),
\]

\[
= 4 (E^2 + |\vec{p}|^2 \cos \theta).
\]

Using these results, we have that

\[
|\mathcal{M}|^2 = \frac{Z^2 e^4}{8|\vec{p}|^4 \sin^4 \theta/2} \left[ E^2 + |\vec{p}|^2 \cos \theta + m^2 \right],
\]

\[
= \frac{Z^2 e^4}{8|\vec{p}|^4 \sin^4 \theta/2} \left[ 2E^2 - |\vec{p}|^2(1 - \cos \theta) \right],
\]

\[
= \frac{Z^2 e^4}{8|\vec{p}|^4 \sin^4 \theta/2} \left[ 2E^2 - 2|\vec{p}|^2 \sin^2 \theta/2 \right],
\]

\[
= \frac{Z^2 e^4 E^2}{4|\vec{p}|^4 \sin^4 \theta/2} \left[ 1 - \left( \frac{|\vec{p}|}{E} \right)^2 \sin^2 \theta/2 \right],
\]

\[
= \frac{Z^2 e^4}{4\beta^2|\vec{p}|^4 \sin^4 \theta/2} \left[ 1 - \beta^2 \sin^2 \theta/2 \right].
\]

In the last two lines we have used the fact that \( \vec{p}/E = \beta \). Now, we showed in Homework 8 that

\[
\frac{d\sigma}{d\Omega} = |\mathcal{M}|^2 = \frac{Z^2 \alpha^2}{16\pi^2}.
\]

Using the fine structure constant to simplify notation, where \( \alpha^2 = \frac{e^4}{16\pi^2} \), it is clear that

\[
\therefore \frac{d\sigma}{d\Omega} = \frac{Z^2 \alpha^2}{4\beta^2|\vec{p}|^2 \sin^4 \theta/2} \left[ 1 - \beta^2 \sin^2 \theta/2 \right].
\]

**Helicity Amplitudes in Yukawa Theory**

We are to consider the amplitude given by,

\[
i\mathcal{M} = \frac{p'}{p} \frac{k'}{k} + \frac{p}{p'} \frac{k'}{k'}
\]

\[
= (\vec{q} \cdot \vec{p}) u(p) \frac{1}{(p-p')^2 - m_5^2} \bar{u}(k') u(k) - \bar{u}(p') u(k) \frac{1}{(p'-k')^2 - m_5^2} \bar{u}(k') u(p).
\]

a) We are to derive the selection rules for helicity for this theory.

We can best understand the selection rules by requiring that one of the spinors is in a projection. To bring the projection operator to the neighboring spinor (in either diagram and starting from any outside term) requires that the projection anticommutes through a \( \gamma^0 \). Therefore, the interaction must flip the spins. Exempli Gratia, \( \bar{u} \gamma^5 u_R = u^\dagger \gamma^0 \gamma^5 u_R = \bar{u}L u_R \).

b) Given these selection rules, what are the non-vanishing amplitudes? These are the only possible terms that involve both incoming states flipping their spin in the outgoing states. So, the nonzero amplitudes are \( \mathcal{M}_{LL,RR}, \mathcal{M}_{RR,LL}, \mathcal{M}_{LR,RL}, \mathcal{M}_{RL,LR}, \mathcal{M}_{RL,LL}, \mathcal{M}_{LR,RR} \).

c) We are to use problem 5 of Homework 5 to compute the explicit form of the two-spinors. We should use this to find the eigenvectors \( u_\lambda(p) \) at very high energies. This is a relatively straight forward calculation. We derived quite some time ago that in the high energy limit for general