Similar to our computation above, to find $\beta_\rho$ we must compute the renormalization counter-term $\delta_\rho$.

To the one-loop order, we can find $\delta_\rho$ by computing,

\[
i M = -i(\rho/3) + (-i\lambda)(-i\rho/3) [V(t) + V(t)] + (-i\rho/3)^2 [2V(u) + 2V(s)] - i\delta_\rho/3.
\]

Recall that we have already computed the divergence of the function $V(k)$ and noted that it was independent of $k$. Therefore,

\[
i\delta_\rho/3 = (i\lambda)(-i\rho/3) [V(t) + V(t)] + (-i\rho/3)^2 [2V(u) + 2V(s)],
\]

\[
= -\lambda\rho/3 - i\frac{\lambda^2}{16\pi^2} \log \frac{\Lambda^2}{M^2} - (\rho/3)^2 - i\frac{\lambda^2}{8\pi^2} \log \frac{\Lambda^2}{M^2},
\]

\[
\therefore \delta_\rho = \frac{1}{16\pi^2} [\lambda\rho + 2\rho^2/3] \log \frac{\Lambda^2}{M^2}.
\]

Because there are no divergent self-energy diagrams in this theory to one-loop order, we have that the $\beta$-function for $\rho$ is given precisely by twice the coefficient of the log divergence in $\delta_\rho$,

\[
\therefore \beta_\rho = \frac{1}{8\pi^2} [\lambda\rho + 2\rho^2/3].
\]

(1.b.2)

Let us now consider the $\beta$-function associated with the ration $\lambda/\rho$. Using the chain rule for differentiation and the definition of the general $\beta$-function, we see that

\[
\beta_{\lambda/\rho} = \frac{1}{\rho^2} \beta_{\lambda\rho} - \beta_\rho \lambda = \frac{1}{\rho^2} \left[ \frac{3\lambda^2\rho}{16\pi^2} + \rho^3 \frac{3\lambda^2}{48\pi^2} - \frac{\rho^2\lambda}{8\pi^2} - \frac{\rho^2\lambda}{12\pi^2} \right],
\]

\[
= \frac{(\lambda/\rho)^2\rho}{16\pi^2} + \rho \frac{\rho}{48\pi^2} - \frac{(\lambda/\rho)}{12\pi^2},
\]

\[
= \frac{\rho}{48\pi^2} \left[ 3(\lambda/\rho)^2 - 4(\lambda/\rho) + 1 \right],
\]

\[
\therefore \beta_{\lambda/\rho} = \frac{\rho}{48\pi^2} \left( 3\lambda/\rho - 1 \right) (\lambda/\rho - 1).
\]

(1.c.1)

We see immediately that the two roots of $\beta_{\lambda/\rho}$ occur when $\lambda/\rho = 1, \frac{1}{3}$ and because the second derivative of $\beta_{\lambda/\rho}$ is $6 > 0$, we know that $\beta_{\lambda/\rho} < 0$ for $\lambda/\rho \in (\frac{1}{3},1)$ and $\beta_{\lambda/\rho} > 0$ for $\lambda/\rho > 1$. Therefore, for all $\lambda/\rho > \frac{1}{3}$, $\lambda/\rho$ will flow to $\lambda/\rho = 1$. See Figure 1 below.

Therefore at large distances the couplings will flow to $\lambda = \rho$. This introduces a continuous $O(2)$ symmetry into the theory. To see this, let us define $\varphi \equiv \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$. In this notation, the Lagrangian simply reads

\[
\mathcal{L} = \frac{1}{2} \left( \partial_\mu \varphi \right)^2 - \frac{\lambda}{4!} \varphi^4.
\]

(1.e.1)

This Lagrangian is clearly invariant to $O(2)$ transformations which correspond to changing the phase of $\varphi$.

\[\text{Figure 1. Renormalization Group Flow as a function of scale. Arrows show } p \rightarrow 0 \text{ flow.}\]