TQFTs in Context

(Bradley Zykoski, Student Geometry/Topology Seminar)

Physical Motivation

- In the context of physics, TQFTs represent a partial attempt to address the problem of quantum gravity. On this subject, I can do no better than to quote John Baez: [Baez, 1999]

  "We have, not one, but two fundamental theories of the physical universe: general relativity, and the Standard Model of particle physics based on quantum field theory. The former takes gravity into account but ignores quantum mechanics, while the latter takes quantum mechanics into account but ignores gravity. In other words, the former recognizes that spacetime is curved but neglects the uncertainty principle, while the latter takes the uncertainty principle into account but pretends that spacetime is flat. Both theories have been spectacularly successful in their own domain, but neither can be anything more than an approximation to the truth. Clearly some synthesis is needed: at the very least, a theory of quantum gravity, which might or might not be part of an overarching 'theory of everything.' Unfortunately, attempts to achieve this synthesis have not yet succeeded."

- Topological quantum field theory (TQFT) is a model of quantum field theory that does not pretend that spacetime is flat: it allows spacetime to have nontrivial global topology. However, it is not a complete solution to the problem of quantum gravity, since

TQFT <u>completely ignores</u> any local geometry (curvature, length, etc.) of spacetime! I would like to remark how wonderful it is that even a partial solution to a problem in physics ~~produces~~ produces such interesting mathematics.

<u>Mathematical Construction</u>

- If we were to freeze time at some t_0, space would be a 3-manifold Σ_0. Freezing time again at some $t_1 > t_0$ gives us another 3-manifold Σ_1, and the spacetime in between Σ_0 and Σ_1 is a 4-manifold M with boundary $\partial M = \Sigma_0 \cup \Sigma_1$. We generalize this as follows:

- <u>Defn:</u> (In- and out-boundary) Given an oriented n-manifold M and a connected oriented $(n-1)$-manifold N with an embedding $\varphi : N \hookrightarrow M$ such that $\varphi(N)$ is a component of ∂M, we say that N $(\equiv \varphi(N))$ is an <u>in-boundary component</u> of M if φ reverses orientation, and is an <u>out-boundary component</u> of M if φ preserves orientation.

- <u>Defn:</u> (Cobordism) A <u>cobordism</u> between two closed oriented $(n-1)$-manifolds Σ_0, Σ_1 is a compact oriented n-manifold M with two embeddings

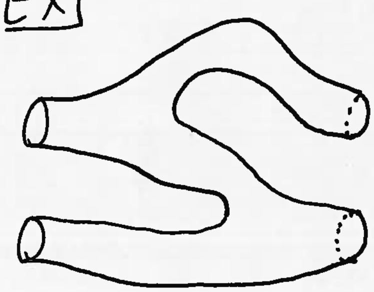
$$\Sigma_0 \hookrightarrow M \hookleftarrow \Sigma_1$$

such that Σ_0 is embedded as the entire in-boundary of M and Σ_1 is embedded as the entire out-boundary of M. Two cobordisms are said to be equivalent if they admit a diffeomorphism $\varphi: M \to M'$ relative to the boundaries, i.e. making the diagram

$$\begin{tikzcd} & M' & \\ \Sigma_0 \arrow[ur] \arrow[dr] & & \Sigma_1 \arrow[ul] \arrow[dl] \\ & M \arrow[uu, "\varphi"] & \end{tikzcd}$$

commute.

· EX



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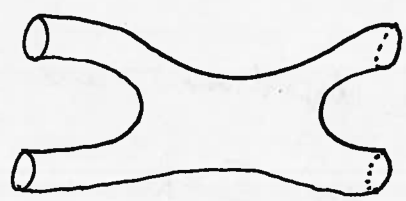
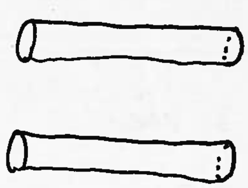


are equivalent cobordisms from $S^1 \sqcup S^1$ to $S^1 \sqcup S^1$, but



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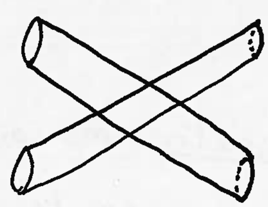


Note: This is not an intersection. There is no ambient space here.

are not, since any diffeomorphism between these latter two induces a swapping of the two components of $S^1 \sqcup S^1$.

· Defn: The category $n\mathrm{Cob}$ is the category whose objects are closed oriented $(n-1)$-manifolds and whose arrows are equivalence classes of cobordisms between them.

- This category is an example of a monoidal category.

- Defn: A monoidal category (or $\otimes$-category) C is a category equipped with a product $\otimes : C \times C \to C$ and a unit object $I \in \mathrm{obj}(C)$ such that $\otimes$ is associative (up to coherent natural isomorphism) and $A \otimes I \cong A \cong I \otimes A$ for every $A \in \mathrm{obj}(C)$ (where these isomorphism are natural and satisfy some coherence conditions). (You may safely ignore any details in parentheses, including this one.)

- EX
    - nCob with $\otimes = \coprod$ (disjoint union) and $I = \emptyset$.
    - Vect_k with $\otimes = \otimes$ and $I = k$.
    - Set with $\otimes = \times$ and $I = \{*\}$.
    - Cat with $\otimes = \times$ and $I = *$ (the terminal category).

- Defn: (Intentionally vague) A symmetric monoidal category is one that has a natural and coherent way to permute the factors of a tensor product, e.g. $A \otimes B \otimes C \cong B \otimes A \otimes C$.

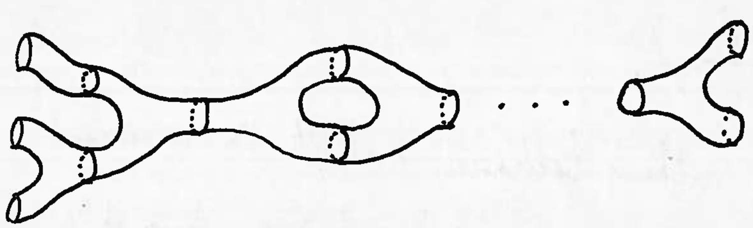
- Defn: A symmetric monoidal functor between symmetric monoidal categories is one that respects their symmetric & monoidal structures.

- Each object of $n\mathrm{Cob}$ is meant to represent physical space, and each arrow is meant to represent spacetime. To each object Σ, a TQFT assigns a vector space $Z(\Sigma)$ called the <u>space of states</u> (very roughly speaking, a <u>state</u> is a probability distribution of possible measurable quantities).
  To each arrow $M : \Sigma_1 \to \Sigma_2$, a TQFT assigns a linear map $Z(M) : Z(\Sigma_1) \to Z(\Sigma_2)$, called the <u>time evolution operator</u> (very roughly speaking, this linear map is meant to express how states evolve with time due to the dynamics of the theory).
  Moreover, a TQFT does this in a functorial way that respects the symmetric monoidal structure of $n\mathrm{Cob}$. Succinctly:

- <u>Defn:</u> (TQFT) A <u>TQFT</u> is a symmetric monoidal functor
$$Z : (n\mathrm{Cob}, \amalg, \emptyset) \longrightarrow (\mathrm{Vect}_k, \otimes, k),$$
where k is some (mathematical!) field.

- <u>Looking at 2Cob</u>

  In 2Cob, every arrow can be decomposed e.g.



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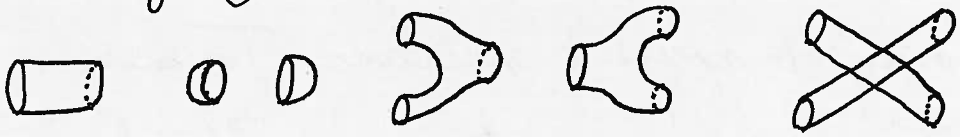
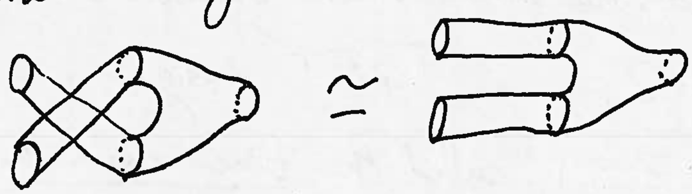
as a gluing of the six elementary cobordisms:



- Relations between these generators are induced by equivalence of cobordisms: e.g.



Frobenius Algebras

- Defn: Let k be a field. A Frobenius algebra A over k is a k-algebra with a liner map (called a pairing)

$$\beta : A \otimes A \longrightarrow k$$
$$v \otimes w \longmapsto \langle v | w \rangle$$

satisfying:

Associativity: $\forall u, v, w \in A$, $\langle vu | w \rangle = \langle v | uw \rangle$

Nondegeneracy: $\exists$ a linear map $\gamma : k \to A \otimes A$ (called a copairing) such that the following compositions are equal to the identity $Id : A \to A$.

$$A \cong A \otimes k \xrightarrow{Id \otimes \gamma} A \otimes A \otimes A \xrightarrow{\beta \otimes Id} A, \quad \text{and}$$

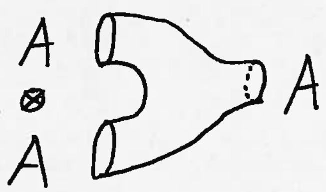
$$A \cong k \otimes A \xrightarrow{\gamma \otimes Id} A \otimes A \otimes A \xrightarrow{Id \otimes \beta} A.$$

- The nondegeneracy condition seems rather technical, but it is equivalent to the condition that $v \mapsto \langle v | \cdot \rangle$ is an isomorphism $A \to A^*$. Hence Frobenius algebras are finite-dimensional.

- The induced form $\varepsilon : A \to k,\ v \mapsto \langle v | 1 \rangle = \langle 1 | v \rangle$ is called the <u>Frobenius form</u>.

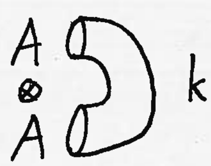
- We introduce "purely formal" notation for the operations in a Frobenius algebra A, though it will quickly become apparent how this notation can be made rigorous:
  - We denote the structural operations for A as follows:

<u>Multiplication $m : A \otimes A \to A$</u>

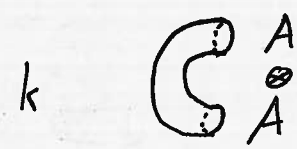


<u>Unit $1 \in A$ (i.e. $\eta : k \to A,\ 1 \mapsto 1$)</u>

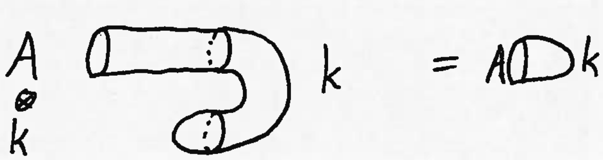
<u>Frobenius pairing $\beta : A \otimes A \to k$</u>



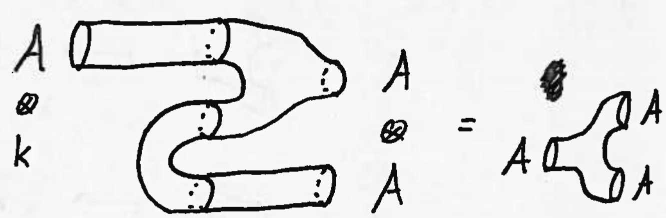
<u>Copairing $\gamma : k \to A \otimes A$</u>



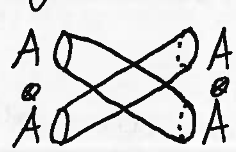
<u>Frobenius form $\varepsilon = \beta \circ (Id \otimes \eta)$</u>



<u>Comultiplication $\Delta = (m \otimes Id) \circ (Id \otimes \gamma)$</u>



<u>Interchange of tensor factors $v \otimes w \mapsto w \otimes v$</u>

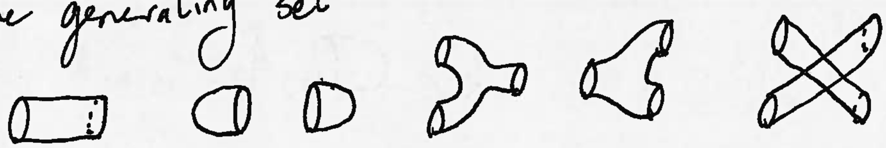


- The idea of this graphical notion suggests the following theorem:

- <u>Theorem</u>: Any commutative Frobenius algebra A can be obtained from a TQFT $Z: 2Cob \to Vect_k$, where $A = Z(S^1)$.

- <u>What must one do to prove this theorem?</u>

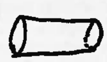
- As remarked previously, the arrows of $2Cob$ are given by the generating set



We must show that under the assignment

$$\emptyset \longmapsto k$$

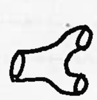
$$S^1 \longmapsto A$$

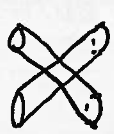
 $\longmapsto Id$

 $\longmapsto \eta$

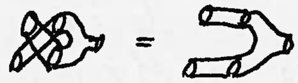
 $\longmapsto \varepsilon$

 $\longmapsto m$

 $\longmapsto \Delta$

 $\longmapsto (v \otimes w \mapsto w \otimes v)$,

all of the relations in $2Cob$ are satisfied by A and its structural operations.

We do not prove this here, but we remark that it is true. As one example, the relation  in 2Cob implies that our Frobenius algebra must be commutative. □

A glimpse of the Cobordism Hypothesis

· A commutative Frobenius algebra, as an object of Vect_k, is defined purely in terms of certain linear structural maps and diagrams they must satisfy. We can therefore generalize this notion to that of a <u>commutative Frobenius object</u> in any arbitrary symmetric monoidal category $\mathcal{C}$, by defining such an object in terms of the same diagrams used to define a commutative Frobenius algebra in the specific case of $\mathcal{C} = \mathrm{Vect}_k$. Let $\mathrm{cFrob}(\mathcal{C})$ denote the category whose objects are commutative Frobenius objects in $\mathcal{C}$ and whose arrows are arrows that respect the commutative Frobenius structure (note that this is deliberately vague!). Further, for symmetric monoidal categories $\mathcal{B}, \mathcal{C}$, let $\mathrm{SymMonCat}(\mathcal{B},\mathcal{C})$ denote the category whose objects are symmetric monoidal functors $\mathcal{B} \to \mathcal{C}$ and whose arrows are natural transformations respecting the monoidal structure (again, this is vague!). The preceding theorem may thereby be upgraded to the following theorem.

· <u>Theorem</u>: There is an equivalence of categories

$$\mathrm{SymMonCat}(2\mathrm{Cob}, \mathcal{C}) \simeq \mathrm{cFrob}(\mathcal{C})$$

for any symmetric monoidal category $\mathcal{C}$, which is given on objects by $Z \mapsto Z(S^1)$.

· For the sake of generalization, the most important thing to note about Frobenius objects is that their pairing induces a <u>duality</u> (recall $A \cong A^*$ for Frobenius algebras, given by $v \mapsto \langle v|\cdot\rangle$). When generalizing to the Cobordism Hypothesis, one considers what are called <u>fully dualizable</u> objects.

· The generalization to the Cobordism Hypothesis requires us to leave behind the notion of <u>category</u> for the more general notion of <u>n-category</u>. To be (perhaps irresponsibly) brief, an n-category is like a category, except there are also arrows between arrows, arrows between these, and so on. In doing this, we must replace the category $n\mathrm{Cob}$ with an n-category of (framed, extended) cobordisms called $n\mathrm{Cob}^{fr}_{ext}$, whose objects are 0-manifolds, arrows are cobordisms between these, and so on. If we let $\mathrm{Dual}(\mathcal{C})$ denote the category of fully dualizable objects in an n-category, then we can state the Cobordism Hypothesis:

· <u>Cobordism Hypothesis</u>: Let C be a symmetric monoidal n-category. Then we have a correspondence

$$\mathrm{SymMonCat}(n\mathrm{Cob}^{fr}_{ext}, C) \simeq \mathrm{Dual}(C),$$

given by $Z \mapsto Z(\text{point})$.

☆ As I have not defined nearly any of the terms used here, the key takeaway should really just be that the Cobordism Hypothesis appears similar in form to the previous theorem, most of whose terms I have actually defined, and which is proved in [Kock, 2003].

· The Cobordism Hypothesis was proposed by Baez and Dolan in 1995 and proved by Lurie (perhaps modulo the elucidation of some technical details?) in 2009.

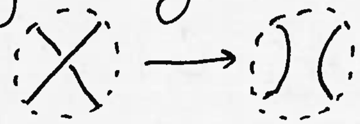
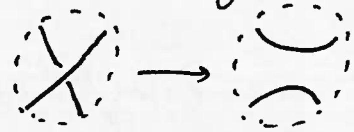
<u>Applications</u>

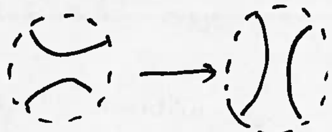
· Note that a cobordism in $n\mathrm{Cob}$ from $\emptyset$ to $\emptyset$ is precisely a closed n-manifold M. Appling a TQFT Z gives

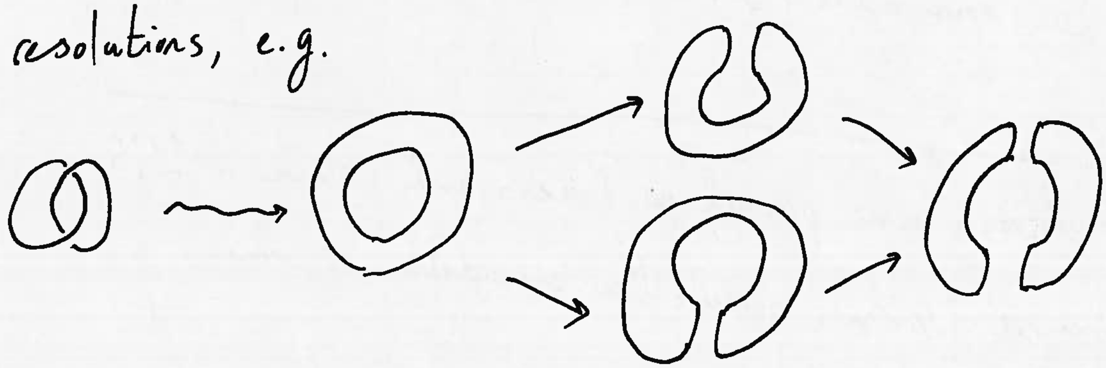
$$Z(M : \emptyset \to \emptyset) = Z(M) : Z(\emptyset) = k \longrightarrow k = Z(\emptyset).$$

But a linear map is nothing more, in the case $k \to k$, than an element of k! Hence any TQFT gives us a diffeomorphism invariant of n-manifolds.

or link!

- One may resolve any crossing  in a knot diagram in one of two ways: , or .

Hence a knot diagram with N crossings has 2^N possible resolutions. If we arrange these resolutions in a diagram with an arrow pointing from one resolution R_0 to another R_1 if R_1 may be obtained from R_0 via some change of resolution of the form , then we get a diagram of resolutions, e.g.



Each node of this diagram is a disjoint union of circles, and any two nodes related by an arrow differ by one circle. Hence we can label each arrow with either the cobordism  or .

Applying a TQFT to this diagram gives a diagram in Vect_k. Each square of this diagram commutes, and so multiplying precisely one arrow of each square by -1 gives an anti-commuting diagram. If we let V_i be the direct sum of all vector spaces given by resolutions with precisely i resolutions of the form , then the induced maps $d_i : V_i \to V_{i-1}$, i.e. the sums of arrows between the summands of V_i and the summands of V_{i-1}, form a chain complex

$$\cdots \xrightarrow{d_{i+2}} V_{i+1} \xrightarrow{d_{i+1}} V_i \xrightarrow{d_i} V_{i-1} \xrightarrow{d_{i-1}} \cdots ,$$

because each square of the diagrams was anti-commutative. The homology of this chain complex is called the <u>Khovanov homology</u> of the knot (or link!) that we started with, and it is an invariant of this knot (or link!).

<u>References</u>

<u>Primary References:</u>

- For the physical motivation: Baez, John. <u>Higher-Dimensional Algebra and Planck-Scale Physics</u>. 1999. Published 2001 in <u>Physics Meets Philosophy at the Planck Scale</u>, eds. Callender & Huggett, Cambridge U Press.

- For the physical motivation and some very lucid discussion of category theory: Bartlett, Bruce. <u>Categorical Aspects of Topological Quantum Field Theories.</u> Masters thesis in Theoretical Physics, Utrecht University, 2005. arXiv:math/0512103.

- For all of the mathematical content up to the penultimate theorem: Kock, Joachim. <u>Frobenius Algebras and 2D Topological Quantum Field Theories (short version)</u>. url: mat.uab.es/~kock/TQFT/FS.pdf. There is also a

longer version in the form of a book, published in 2003.

- For the imprecise but hopefully somewhat enlightening statements concerning the Cobordism Hypothesis: Video lectures given by Jacob Lurie. Part 1 is at www.youtube.com/watch?v=Bo8GNfN-Xn4. (4 parts)
- For the precise statement of the Cobordism Hypothesis and its proof: Lurie, Jacob. On the Classification of Topological Field Theories. arXiv:0905.0465.
- For the construction of Khovanov Homology: Bar-Natan, Dror. Khovanov's homology for tangles and cobordisms. Geometry & Topology, Vol. 9 (2005), pp. 1443–1499.

· Other Good References:

- For a discussion of higher algebra and the original proposition of the Cobordism Hypothesis: Baez, John and Dolan, James. Higher-dimensional Algebra and Topological Quantum Field Theory. J. Math. Phys. 36 (1995), 6073–6105.
- For Atiyah's original definition of TQFT: Atiyah, Michael. Topological Quantum Field Theories. Publications Mathématiques de l'I.H.É.S., Vol. 68 (1988), pp. 175-186.
- For a brief introduction to Khovanov Homology: Bar-Natan, Dror. On Khovanov's categorification of the Jones polynomial. Algebraic & Geometric Topology. Vol. 2 (2002), pp. 337–370.