Spin Structures

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February 4, 2020

Outline

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§1 Definition of the spin groups

- Definition: Let $n \geq 2$. We define $Spin(n)$ to be the unique double cover of $SO(n)$. In fact, when $n \geq 3$, we will see that $Spin(n)$ is in fact the universal cover of $SO(n)$.

- It remains to see whether this definition makes sense. Since $SO(2) \cong S^1$, and $S^1 \to S^1$, $e^{i\theta} \mapsto e^{2i\theta}$ is the unique double cover, we see that $Spin(2) \cong S^1$.

- We will return to $SO(3)$ soon, but first we will show that $\pi_1(SO(n)) = \pi_1(SO(3))$ for $n \geq 3$.

- We have a fiber bundle

$$\begin{array}{ccc} SO(n) & \to & SO(n+1) \\ & & \downarrow \\ & & S^n \end{array}$$

because $SO(n+1)$ acts transitively on S^n by isometries, with point-stabilizer $SO(n)$.

- Now, we have a long exact sequence of homotopy groups given by this fibration:

$$\cdots \to \pi_k(SO(n)) \to \pi_k(SO(n+1)) \to \pi_k(S^n) \to \pi_{k-1}(SO(n)) \to \cdots$$

- Since $n \geq 3$, we have $\pi_2(S^n) = \pi_1(S^n) = 0$, and so part of our long exact sequence is:

$$0 = \pi_2(S^n) \to \pi_1(SO(n)) \to \pi_1(SO(n+1)) \to \pi_1(S^n) = 0$$

Since this sequence is exact, it follows that $\pi_1(SO(n)) \cong \pi_1(SO(n+1))$ for $n \geq 3$.

- It remains to compute $\pi_1(SO(3))$. Recall that every $A \in SO(3)$ has an axis of rotation $\text{axis}(A)$ whose orthogonal complement is some plane $\text{plane}(A)$. A choice of unit vector in $\text{axis}(A)$ induces an orientation on $\text{plane}(A)$, and hence a notion of clockwise vs. counterclockwise rotation. Let v_A be the unit vector in $\text{axis}(A)$ that orients $\text{plane}(A)$ so that A acts on $\text{plane}(A)$ by a rotation of at most π. Then v_A is well-defined except when the rotation is precisely equal to π, in which case there are two choices for v_A, differing by a sign.

Furthermore notice that the pair (v_A, angle θ_A of rotation in $\text{plane}(A)$) entirely determines $A \in SO(3)$. Let $\mathbb{B}^3$ denote the unit ball in $\mathbb{R}^3$. It then follows that we have a homeomorphism

$$SO(3) \xrightarrow{\sim} \mathbb{B}^3/(v \in \partial \mathbb{B}^3 \sim -v) \cong \mathbb{RP}^3$$

given by $A \mapsto \left[\frac{\Theta_A}{\pi} v_A\right]_\sim$. Therefore $\pi_1(SO(3)) = \pi_1(\mathbb{R}P^3) = \mathbb{Z}/2\mathbb{Z}$.

- We conclude that $\pi_1(SO(n)) \cong \mathbb{Z}/2\mathbb{Z}$ for every $n \geq 3$, and so each $SO(n)$ has a unique double cover, namely its universal cover. We call this covering $\mathrm{Spin}(n)$, and it has the structure of a Lie group induced by the covering $\mathrm{Spin}(n) \longrightarrow SO(n)$.

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§2 Generalities on principal bundles

- Guiding principle: A principal bundle is a blueprint for a general fiber bundle.

- Definition: A principal G-bundle for a topological group G (usually we will consider G a Lie group) is a fiber bundle $E \xrightarrow{\pi} B$, along with:

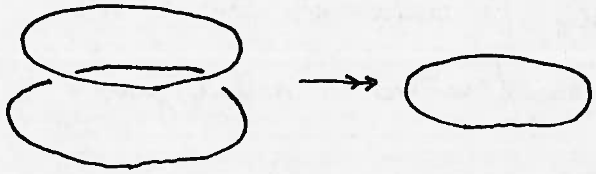
  (1) the additional data of a right group action $G \curvearrowright E$ by homeomorphisms that preserves the fibers of π:
  $$\pi(x.g) = \pi(x) \qquad \forall x \in E,\ \forall g \in G,$$

  (2) the additional constraint that the induced action $G \curvearrowright \pi^{-1}(b)$ on the fiber over every $b \in B$ is free (i.e. has no fixed points) and transitive. Notice that it follows that for every $x_0 \in \pi^{-1}(b)$, the orbit map $G \to \pi^{-1}(b)$, $g \mapsto x_0 \cdot g$ is a homeomorphism.

- If G is a Lie group, we will usually require all the involved maps to be smooth, and the word "homeomorphism" above will be replaced with "diffeomorphism."

Examples of principal bundles:

(1) Up to isomorphism, there are only two principal $(\mathbb{Z}/2\mathbb{Z})$-bundles over S^1:

(a)  , where $\mathbb{Z}/2\mathbb{Z}$ acts by swapping the components,

(b) $$S^1 \xrightarrow{e^{i\theta} \mapsto e^{2i\theta}} S^1,$$ where $1 \in \mathbb{Z}/2\mathbb{Z}$ acts by $e^{i\theta} \longmapsto e^{i(\theta+\pi)}$.

(2) Let $\begin{array}{c} V \to E \\ \downarrow \\ B \end{array}$ be a vector bundle, and let

$$GL(E) = \{(b, \mathcal{B}) \mid b \in B,\ \mathcal{B} \text{ is a basis of } \pi^{-1}(b)\}.$$

Then we have an action $GL_n(\mathbb{R}) \curvearrowright E$, where $n = \dim(V)$, given by

$$(b, \mathcal{B}).A = (b, A^{-1}\mathcal{B}) \quad \text{for every } A \in GL_n(\mathbb{R}).$$

This action is clearly free and transitive on the fibers of

$$\begin{aligned} GL(E) &\longrightarrow B \\ (b, \mathcal{B}) &\longmapsto b, \end{aligned}$$

and hence $GL(E) \to B$ is a principal $GL_n(\mathbb{R})$-bundle.

This is called the <u>frame bundle</u> of E.

(3) Let $Y \to X$ be a normal covering, with deck group D. It is clear from the definition that $Y \to X$ is a principal D-bundle. In particular, the universal covering $\widetilde{X} \to X$ is a principal $\pi_1(X)$-bundle.

• Let us return to the guiding principle at the beginning of the section. We want to <u>construct</u> more general fiber bundles out of principal bundles.

• <u>Definition</u>: Let G act on a space X by homeomorphisms. Then given any principal G-bundle $E \to B$, we have an action $G \curvearrowright E \times X$ given by $g.(p, x) := (p.g^{-1}, g.x)$ for every $g \in G$. Since $G \curvearrowright E$ preserves the fibers of π, we have an induced map $(E \times X)/G \xrightarrow{\widetilde{\pi}} B$.

The space $(E \times X)/G$ is called an <u>associated G-bundle</u>, and G is said to be its <u>structure group</u>. Since G acts freely and transitively on the fibers of $E \xrightarrow{\pi} B$, if we are given a choice of $p_0 \in \pi^{-1}(b)$, then every $y \in \widetilde{\pi}^{-1}(b)$ can be uniquely expressed in the form

$$y = (p_0, x).$$

Therefore the map $y \mapsto x$ gives a homeomorphism $\widetilde{\pi}^{-1}(b) \to X$.

We conclude that we have a fiber bundle

$$\begin{array}{ccc} X & \longrightarrow & (E \times X)/G \\ & & \downarrow \\ & & B. \end{array}$$

Examples of associated bundles:

(1) Let $E_0 \to S^1$ be the trivial $(\mathbb{Z}/2\mathbb{Z})$-bundle over S^1 in example (1a) above, and let $E_1 \to S^1$ be the nontrivial $(\mathbb{Z}/2\mathbb{Z})$-bundle over S^1 in example (1b) above. Consider the following $(\mathbb{Z}/2\mathbb{Z})$-actions:

- $\mathbb{Z}/2\mathbb{Z} \curvearrowright S^1$ where 1 acts by $e^{i\theta} \mapsto e^{-i\theta}$,
- $\mathbb{Z}/2\mathbb{Z} \curvearrowright [0,1]$ where 1 acts by $x \mapsto 1-x$.

Then:

- $(E_0 \times S^1)/(\mathbb{Z}/2\mathbb{Z})$ is a torus.
- $(E_1 \times S^1)/(\mathbb{Z}/2\mathbb{Z})$ is a Klein bottle.
- $(E_0 \times [0,1])/(\mathbb{Z}/2\mathbb{Z})$ is a cylinder.
- $(E_1 \times [0,1])/(\mathbb{Z}/2\mathbb{Z})$ is a Möbius band.

(2) Let $E \to B$ be a vector bundle, and let $GL(E) \to B$ be its frame bundle. Then for the standard action $GL_n(\mathbb{R}) \curvearrowright \mathbb{R}^n$ where $n = \mathrm{rank}(E)$, we have

$$\left(GL(E) \times \mathbb{R}^n\right) / GL_n(\mathbb{R}) \cong E.$$

(3) Let $\mathbb{R} \to S^1$ be the universal covering, and let $S^1 \times \mathbb{Z} \to S^1$ be the first coordinate projection. Let $f : X \to X$ be a homeomorphism of some space X, and let $\mathbb{Z} \curvearrowright X$ where 1 acts by $f : X \to X$. Then

$$\left((S^1 \times \mathbb{Z}) \times X\right) / \mathbb{Z} = S^1 \times X, \text{ and}$$

$$(\mathbb{R} \times X)/\mathbb{Z} = \text{the mapping torus of } f.$$

· Notice that, with some of these constructions, we did not "use the group to its full potential." Notice that the associated bundles that turned out to be trivial bundles (i.e. Cartesian products) didn't really use the group in any meaningful way at all! Furthermore, if $TM \to M$ is the tangent bundle of a manifold, we could recover TM from $GL(TM)$ as above, or we could place a Riemannian metric on M, define the principal $O(n)$-bundle $O(TM) = \{(b, \mathcal{B}) \mid b \in M, \mathcal{B}$ an orthonormal basis for $T_b M\}$, and get

$$\left(O(TM) \times \mathbb{R}^n\right) / O(n) \cong TM.$$

Again, we didn't really <u>need</u> the whole group $GL_n(\mathbb{R})$ to "build" TM: we can build TM using the smaller group $O(n)$.

- To make these observations precise, we come to the notion of a <u>reduction of structure group</u>.

- <u>Definition</u>: Let G and H be topological groups, with an inclusion $\iota: H \hookrightarrow G$. If we have a principal G-bundle $E \to B$ and a principal H-bundle $Q \to B$, then an embedding $\varphi: Q \to E$ is called a <u>reduction of structure group</u> if:

  (i) The diagram
$$\begin{tikzcd} Q \arrow[rr, "\varphi"] \arrow[dr] & & E \arrow[dl] \\ & B & \end{tikzcd}$$
commutes (i.e. φ preserves fibers)

  (ii) φ is H-equivariant with respect to the inclusion $\iota: H \hookrightarrow G$:
$$\varphi(q \cdot h) = \varphi(q) \cdot \iota(h) \qquad \forall q \in Q,\ h \in H.$$

- <u>Examples of reductions of structure groups</u>:

  - In essence, we have seen all these examples, we just haven't called them reductions in quite this way.

(1) If we ever have a trivial principal G-bundle $B \times G \to B$, then the inclusion of the trivial group $\iota : \{1\} \hookrightarrow G$ and the trivial $\{1\}$-bundle $B \times \{1\} \to B$ give a reduction of structure group

$$\begin{tikzcd} B \times \{1\} \arrow[rr, hook, "\sigma"] \arrow[dr] & & B \times G \arrow[dl] \\ & B & \end{tikzcd}$$

in the obvious way. Notice also that this realizes $B \times G$ as an associated $\{1\}$-bundle via

$$\big((B \times \{1\}) \times G\big)/\{1\} \cong B \times G.$$

Since $B \times \{1\} \cong B$, we see that a principal G-bundle $E \to B$ is trivial iff there is a section $E \underset{\sigma}{\overset{\curvearrowleft}{\longrightarrow}} B$.

(2) As we remarked on page 7, there ought to be a reduction of structure group $O(TM) \hookrightarrow GL(TM)$. Indeed, this is simply the inclusion of $O(TM)$ as a subset of $GL(TM)$.

This construction requires us to know that every smooth manifold M admits a Riemannian metric. Indeed, a bit more playing around with associated bundles reveals that admitting a Riemannian metric is equivalent to having a well-defined reduction $O(TM) \hookrightarrow GL(TM)$, and that such a reduction always exists because $GL_n(\mathbb{R})$ deformation retracts onto $O(n)$, via e.g. the Gram-Schmidt process. This gives a different proof from the commonly cited one, which uses the Whitney embedding theorem to give $M \hookrightarrow \mathbb{R}^N$, and then restricts the Euclidean Riemannian metric on $\mathbb{R}^N$ to the submanifold M.

§3 Spin structures

- You might rightly complain that the definition of reduction of structure group is arbitrarily restrictive — why must the map $H \to G$ be an inclusion? Well, perhaps then the word "reduction" would not be a good choice, but this extramathematical concern notwithstanding, we can let $H \to G$ be more than just an inclusion. In particular, we have:

- Definition (Spin structure): Let M be a Riemannian manifold, and let $\rho : Spin(n) \to SO(n)$ be the double covering, where $n = \dim(M)$. Then a spin structure on M is a principal $Spin(n)$-bundle $E \to M$ along with a double covering $E \xrightarrow{\varphi} \underbrace{SO(TM)}_{\text{the space of positively oriented orthonormal bases}}$ so that:

(i) The diagram
$$\begin{tikzcd} E \arrow[r, "\varphi"] \arrow[dr] & SO(TM) \arrow[d] \\ & M \end{tikzcd}$$
commutes

(ii) φ is $Spin(n)$-equivariant with respect to the covering $\rho: Spin(n) \to SO(n)$:

$$\varphi(p.h) = \varphi(p).\rho(h) \qquad \forall p \in E,\ h \in Spin(n).$$

- Our examples gave us a clue about how to think about reductions of structure group: they tell us that the original group we were using was "bigger than necessary." What then about spin structures?

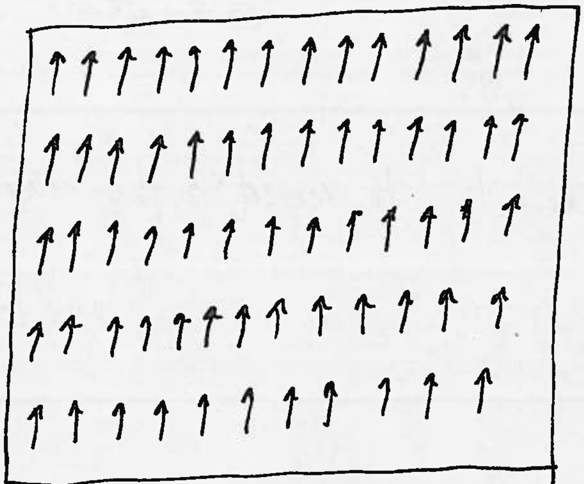
As before, we will try to get some understanding from an example: We consider spin structures on the torus $S^1 \times S^1$ with the standard flat metric.

Since the tangent bundle of the torus is trivial, so is the (positively oriented) orthonormal frame bundle, and so we have $SO(T(S^1 \times S^1)) \cong (S^1 \times S^1) \times SO(2)$
$\cong S^1 \times S^1 \times S^1.$

We can build $S^1 \times S^1 \times S^1$ in a couple different ways, and these differences will be visible from the point of view of the spin structure:

We have a standard parametization $S^1 \times S^1 \times S^1 \cong \{(e^{i\theta}, e^{i\varphi}, e^{i\psi}) \mid \theta, \varphi, \psi \in \mathbb{R}\}$.

There is a vector field given by fixing $\psi = \frac{\pi}{2}$, which looks like:



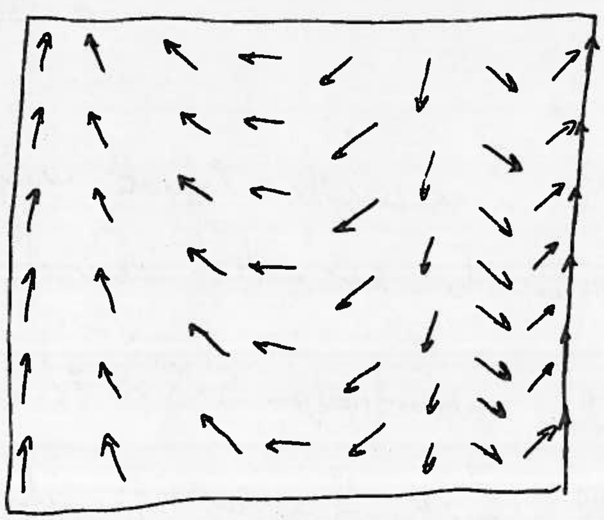
$(\psi = \pi/2)$

We can use this parametrization to build a spin structure:

$S^1 \times S^1 \times (\mathrm{Spin}(2)$ [redacted] $S^1) \longrightarrow S^1$

$(e^{i\theta}, e^{i\varphi}, e^{i\Psi}) \longmapsto (e^{i\theta}, e^{i\varphi}, e^{i2\Psi})$

We can also build a different vector field, which will suggest a different spin structure: the field given by $\psi = \frac{\pi}{2} + \theta$



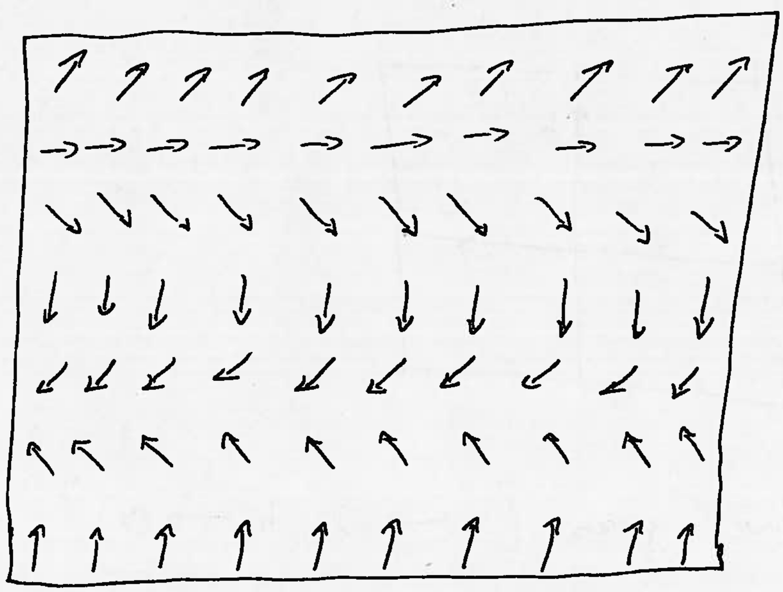
Not a perfect drawing!

$(\psi = \frac{\pi}{2} + \theta)$

This suggests a different spin structure:

$$S^1 \times S^1 \times S^1 \longrightarrow S^1 \times S^1 \times S^1$$
$$(e^{i\theta}, e^{i\varphi}, e^{i\Psi}) \longmapsto (e^{i\theta}, e^{i\varphi}, e^{i(2\Psi + \theta)})$$

Similarly,



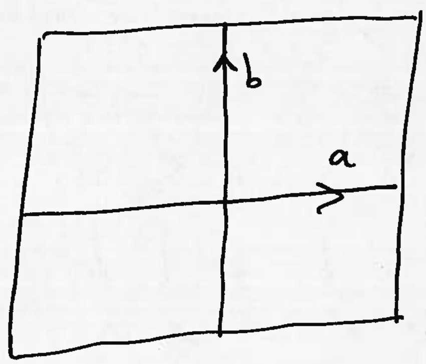
$(\psi = \frac{\pi}{2} + \varphi)$

suggests

$$S^1 \times S^1 \times S^1 \longrightarrow S^1 \times S^1 \times S^1$$
$$(e^{i\theta}, e^{i\varphi}, e^{i\Psi}) \longmapsto (e^{i\theta}, e^{i\varphi}, e^{i(2\Psi + \varphi)}).$$

We can see very explicitly that these spin structures are different by considering how they act on simple closed curves on the torus.

· We will have to elide a discussion of holonomy on principal bundles, but suffice it to say that given a simple closed curve on the torus, its holonomy with respect to one of these spin structures is precisely equal to the mod 2 winding number of the respective vector field along the curve. Hence, for the homology basis



the first spin structure gives $(a \mapsto 0,\ b \mapsto 0)$,
the second structure gives $(a \mapsto 1,\ b \mapsto 0)$,
and the third structure gives $(a \mapsto 0,\ b \mapsto 1)$.

· This illustrates the following general fact:

* So long as spin structures do exist! Some manifolds don't have any!

· Theorem: On a Riemannian manifold* M, there is a (non-canonical) one-to-one correspondence

$$\{\text{spin structures on } M\} \longleftrightarrow H^1(M; \mathbb{Z}/2\mathbb{Z}).$$