Retracting the moduli space of curves

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Outline: 1. What are the Teichmüller and moduli spaces?
2. A cell decomposition for the Teichmüller space
3. A retract for the cell complex

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§1 What are the Teichmüller and moduli spaces?

Let F denote a topological surface of genus g, with n marked points. By the classification of closed oriented surfaces, F is determined uniquely up to homeomorphism by the parameters (i.e. moduli) g and n. Note that these moduli are discrete.

Now consider a Riemann surface X homeomorphic to F. If we give X by some polynomial that it satisfies, then we can vary the coefficients of the polynomial to get new Riemann surfaces not biholomorphic to R. In this way, we see that Riemann surfaces of genus g are determined by continuous moduli.

The question therefore arises whether we can come up with some space of parameters (i.e. moduli space) that parametrizes all Riemann surfaces of genus g with n marked points (or n punctures, if you prefer). Indeed we can, though we will not describe its topology:

$$\mathcal{M}_{g,n} := \left\{ X \;\middle|\; \begin{array}{l} X \text{ is a Riemann surface of genus } g \text{ with} \\ n \text{ marked points/punctures} \end{array} \right\}$$

We will remark, though, that $\mathcal{M}_{g,n}$ has dimension $6g-6+2n$ as a topological space. That is, we can determine a Riemann surface of genus g with n marked points using $6g-6+2n$ real parameters, or $3g-3+n$ complex parameters.

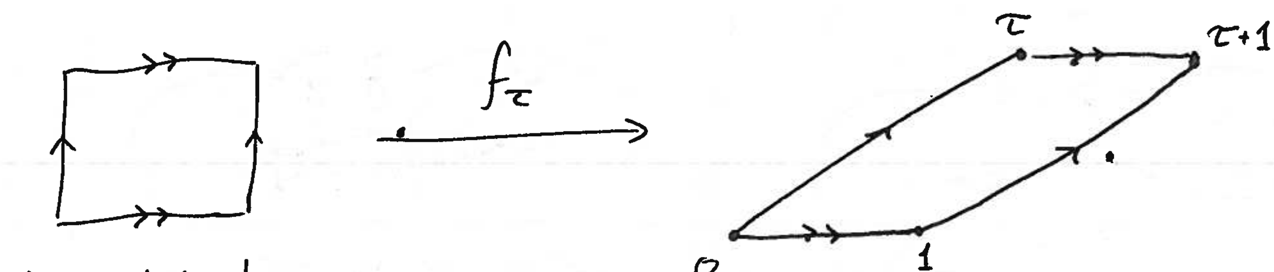
As is often the case in topology, we will find it more convenient to work with the universal cover of $\mathcal{M}_{g,n}$. This universal cover is called the *Teichmüller space*, after Oswald Teichmüller, who introduced techniques with quasiconformal maps to study moduli of Riemann surfaces. It is unfortunate that naming something after a person is typically taken as an honor, because Teichmüller deserved no honors: he was a committed Nazi who antagonized his Jewish colleagues, and died on the front lines fighting for the Third Reich. We nevertheless denote:

$$T_{g,n} := \left. \left\{ (X,f) \;\middle|\; \begin{array}{l} X \text{ is a Riemann surface of genus } g \text{ with} \\ n \text{ marked points/punctures, and} \\ f: F \to X \text{ is an orientation-preserving} \\ \text{homeomorphism} \end{array} \right\} \right/ \sim$$

where $(X,f)\sim(X,g)$ if $g\circ f^{-1}: X \to X$ is isotopic to a biholomorphism. The forgetful map $(X,f)\mapsto X$ from $T_{g,n}$ to $\mathcal{M}_{g,n}$ is the universal covering, and the deck group $\mathrm{Mod}_{g,n}$ is called the *mapping class group*. This is not a deck group/universal covering in the ordinary sense: $\mathrm{Mod}_{g,n}$ does have some finite nontrivial point-stabilizers. We therefore say that $\mathrm{Mod}_{g,n}$ is the *orbifold* deck group, and $T_{g,n}$ the *orbifold* universal cover, of the *orbifold* $\mathcal{M}_{g,n}$.

Throughout these notes, we will consider the example where $g=n=1$ to illustrate the abstract constructions:

Any complex torus with one marked point can be written as $\mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$ for some $\tau\in\mathbb{H}^2=\{z\in\mathbb{C}\mid \mathrm{Im}(z)>0\}$. The lattice vectors $1,\tau$ also endow the torus with a marking:



standard topological torus $\mathbb{R}^2/\mathbb{Z}^2$

complex torus $\mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$.

Therefore each point $\tau\in\mathbb{H}^2$ determines a unique point $(\mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z}), f_\tau)$ in $T_{1,1}$, so $T_{1,1}\cong\mathbb{H}^2$. Two such tori are biholomorphic iff their moduli τ and κ in $\mathbb{H}^2$ satisfy the equation

$$\kappa=\frac{a\tau+b}{c\tau+d}\qquad\text{for some }\begin{pmatrix}a&b\\c&d\end{pmatrix}\in PSL_2\mathbb{Z}.$$

Therefore $M_{1,1}=T_{1,1}/Mod_{1,1}\cong\mathbb{H}^2/PSL_2\mathbb{Z}=\mathbb{C}$.

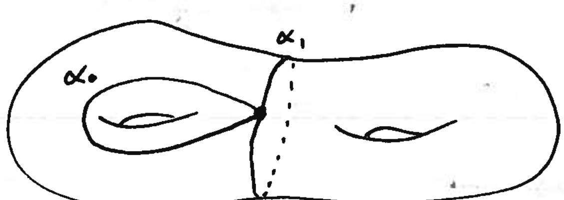
§2 A cell decomposition for the Teichmüller space

Throughout, for simplicity, we will set $n=1$, and so only consider $T_{g,1}$.

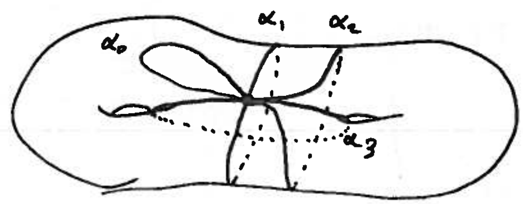
The original idea for the following cell decomposition was due to Thurston, and details for his approach were provided by Bowditch and Epstein. The following presentation perspective via quadratic differentials is due to Mumford. The in these notes is not original: I am following Harer's presentation in [H88] and [H86]. Recall that F is a genus g oriented closed topological surface with $n=1$ marked point $p \in F$.

<u>Definition</u>: A <u>rank k arc system</u> on F is the isotopy class of a collection of $k+1$ simple closed curves $\alpha_0, \dots, \alpha_k$ on F, such that

- $\alpha_i \cap \alpha_j = \{p\}$ when $i \neq j$, and
- no α_i is nullhomotopic, and α_i is not homotopic to α_j when $i \neq j$.



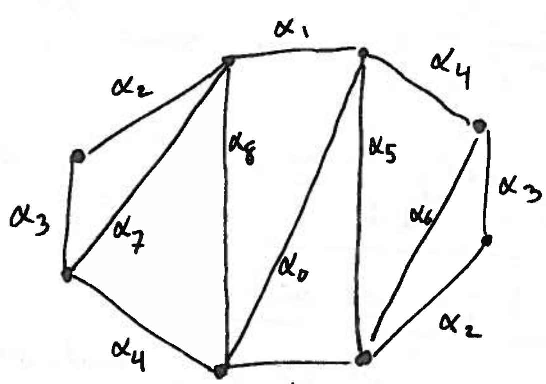
rank 1 arc system on F for $g=2$



not an arc system:

- α_0 is nullhomotopic
- α_1 is homotopic to α_2
- α_3 intersects α_1 & α_2

An arc system can have rank at most $6g-4$, the number of ~~triangles~~ curves (minus one) of a triangulation of F with a single vertex. This is most easily seen by drawing F as a 4g-gon:



Triangulating an octagon $(g=2)$

Let A denote the simplicial complex with a k-simplex for each rank k arc system, where $\langle \beta_0, \dots, \beta_l \rangle$ is considered a face of $\langle \alpha_0, \dots, \alpha_k \rangle$ if $\{\beta_j\}_j \subseteq \{\alpha_i\}_i$. Note that A is $(6g-4)$-dimensional, and points of A are pairs (α, w), where $\alpha = \langle \alpha_0, \dots, \alpha_k \rangle$ is an arc system, and $w = (w_0, \dots, w_k)$ is a tuple of numbers with $\sum w_i = 1$.

An arc system *fills* F if each connected component of $F \setminus \bigcup_i \alpha_i$ is simply connected. Let A_∞ denote the subcomplex of A whose simplices are those arc systems that do *not* fill F.

(Please find attached an excerpt of Harer's notes illustrating A and A_∞ in the case $g = n = 1$.)

Definition: A *meromorphic quadratic differential* is a meromorphic section of the tensor square of the canonical bundle (i.e. holomorphic cotangent bundle) of a Riemann surface.

We will only be considering a very special example: horocyclic quadratic differentials.

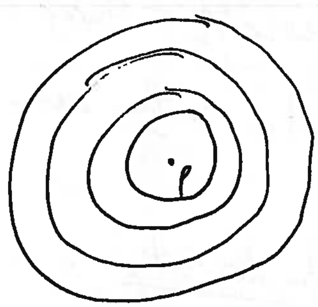
Definition: Let X be a Riemann surface with one marked point p. A *horocyclic quadratic differential* on X is a meromorphic quadratic differential q on X with a single pole, which is order order two and is at the point p. Further, we require that in some coordinates about p, we have

$$q = \frac{c}{z^2} dz^2, \quad c < 0.$$

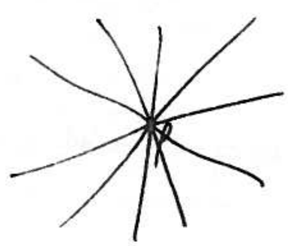
We say that a coordinate system φ is <u>natural</u> for a quadratic differential q if $\varphi^* dz^2 = q$. If we write a horocyclic quadratic differential in the coordinates near p so that $q = \frac{c}{z^2} dz^2$, then <u>natural</u> coordinates for q are given by $\varphi(z) = i\sqrt{c}\log(z)$:

$$\varphi^* dz^2 = \left(\frac{\partial\varphi}{\partial z}\, dz\right) \otimes \left(\frac{\partial\varphi}{\partial z}\, dz\right) = \left(\frac{i\sqrt{c}}{z}\, dz\right) \otimes \left(\frac{i\sqrt{c}}{z}\, dz\right) = \frac{c}{z^2} dz^2 = q.$$

An arc segment on F is called <u>real</u> if it is sent to a horizontal line under φ, and <u>imaginary</u> if it is sent to a vertical line under φ. Therefore, in a neighborhood of p, the real trajectories look like:



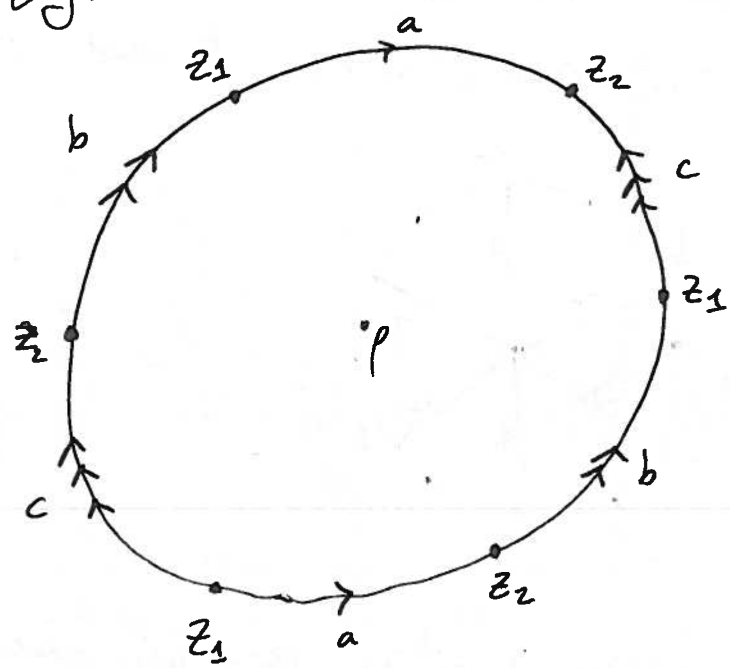
and the imaginary trajectories look like:



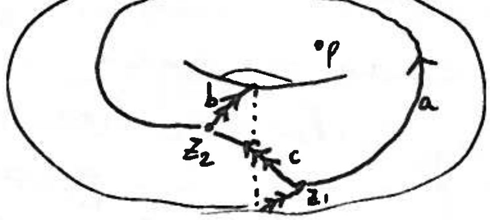
We have introduced these terminologies because of the following result of Strebel:

<u>Theorem (Strebel)</u>: On a Riemann surface X with marked point p, there is a unique horocyclic quadratic differential with a pole at p up to multiplication by positive scalars.

Indeed, Strebel's construction does more than this. It shows that all of the zeroes of q lie on a common real trajectory, so that we have a picture like this:

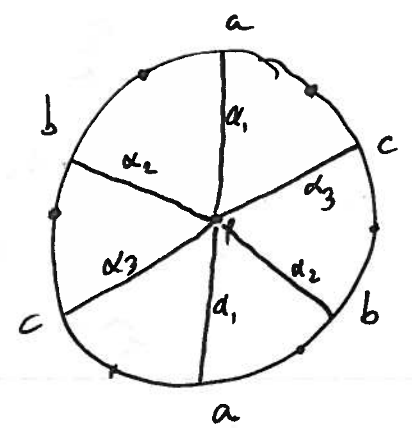


A quadratic differential on a torus with one marked point p, with zeroes z_1, z_2. The arcs a, b, c are real trajectories. Note that our coordinates are 1-1 on the interior of the disk, but on its boundary are not (we have drawn a, b, and c twice each). This is the hexagonal presentation of the torus:



From such a picture, we obtain a point of $A \setminus A_\infty$ as follows:

- Our arc system is made up of the closed loops based at p that run transverse to each of a, b, c:



- Since this arc system is filling, this will indeed give us a point of $A \setminus A_\infty$.
- Each of a, b, c has some Euclidean length in the natural coordinates φ for q. Let w_1, w_2, w_3 be these lengths.
- We now have a pair $(\alpha, w) \in A \setminus A_\infty$.
- (SUBTLETY!): We have defined our arc system on the Riemann surface X on which we have our quadratic differential q. We take the preimage under some homeomorphism $f: F \to X$ to get an arc system on F. If we wanted to be more precise, we would write $(f^{-1}\alpha, w) \in A \setminus A_\infty$.

- We have now described a map

$$\overline{T}_{g,1} \longrightarrow A \setminus A_\infty$$
$$(X,f) \longmapsto (f^{-1}\alpha, w),$$

where α and w are the arc system and set of weights we described above for the unique horocyclic quadratic differential q on X, normalized by a positive constant so that $\sum w_i = 1$.

- Indeed, this map is a homeomorphism, and so we get our desired decomposition of $\overline{T}_{g,1}$ into cells. To be slightly more careful, note that A_∞ is a subcomplex of A, and so $A \setminus A_\infty$ does not actually contain all the faces of its cells; it isn't, strictly speaking, a CW-complex. This is fine; it is nevertheless a useful combinatorial decomposition of $\overline{T}_{g,1}$, as we shall see in the following section.

§3 A retract for the cell complex

Let Y denote the simplicial complex whose k-simplices are arc systems of rank $6g-4-k$ that fill F, where $\langle \alpha_0, \dots, \alpha_{6g-4-k} \rangle$ is a face of $\langle \beta_0, \dots, \beta_{6g-4-l} \rangle$

if $\{\beta_j\}_j \subseteq \{\alpha_i\}_i$. Since the minimal number of arcs based at $p \in F$ that fill F is equal to the number of arcs in a homology basis for F (i.e. $2g$), the dimension d of Y is

$$6g - 4 - d = 2g - 1$$

$$\boxed{d = 4g - 3}.$$

We will see that $M_{g,1}$ has a deformation retraction onto $Y/\mathrm{Mod}_{g,1}$. A priori, $M_{g,1}$ could have nontrivial Betti numbers all the way up until its dimension $6g - 6 + 2 \cdot 1 = 6g - 4$, but this shows that in fact the Betti numbers must all vanish after degree $4g-3$.

This deformation retraction will be performed inductively, and so we will need a slight extension of our definition of an arc system to surfaces with boundary.

<u>Definition</u>: Let S be a compact surface with boundary, and with at least one marked point on every boundary component. A <u>rank k arc system</u> is the isotopy class of $k+1$ simple arcs $\alpha_0, \ldots, \alpha_k$ on F with endpoints in the set of marked points, so that: $\cdot\, \alpha_i \cap \alpha_j = \{p'\}$ for p' a marked point when $i \neq j$, and

- no α_i is nulhomotopic, α_i is not homotopic rel marked points to α_j when $i \neq j$, and no α_i is homotopic rel marked points to a segment of ∂F containing no marked points.

We define $A(S)$ analogously to A.

Recall that the <u>first barycentric subdivision</u> B° of a simplicial complex B has a vertex (of <u>weight</u> k) for every k-simplex of B, and an r-simplex for every length r chain of inclusions

$$\beta_0 \subseteq \cdots \subseteq \beta_r$$

of simplices β_i of B:

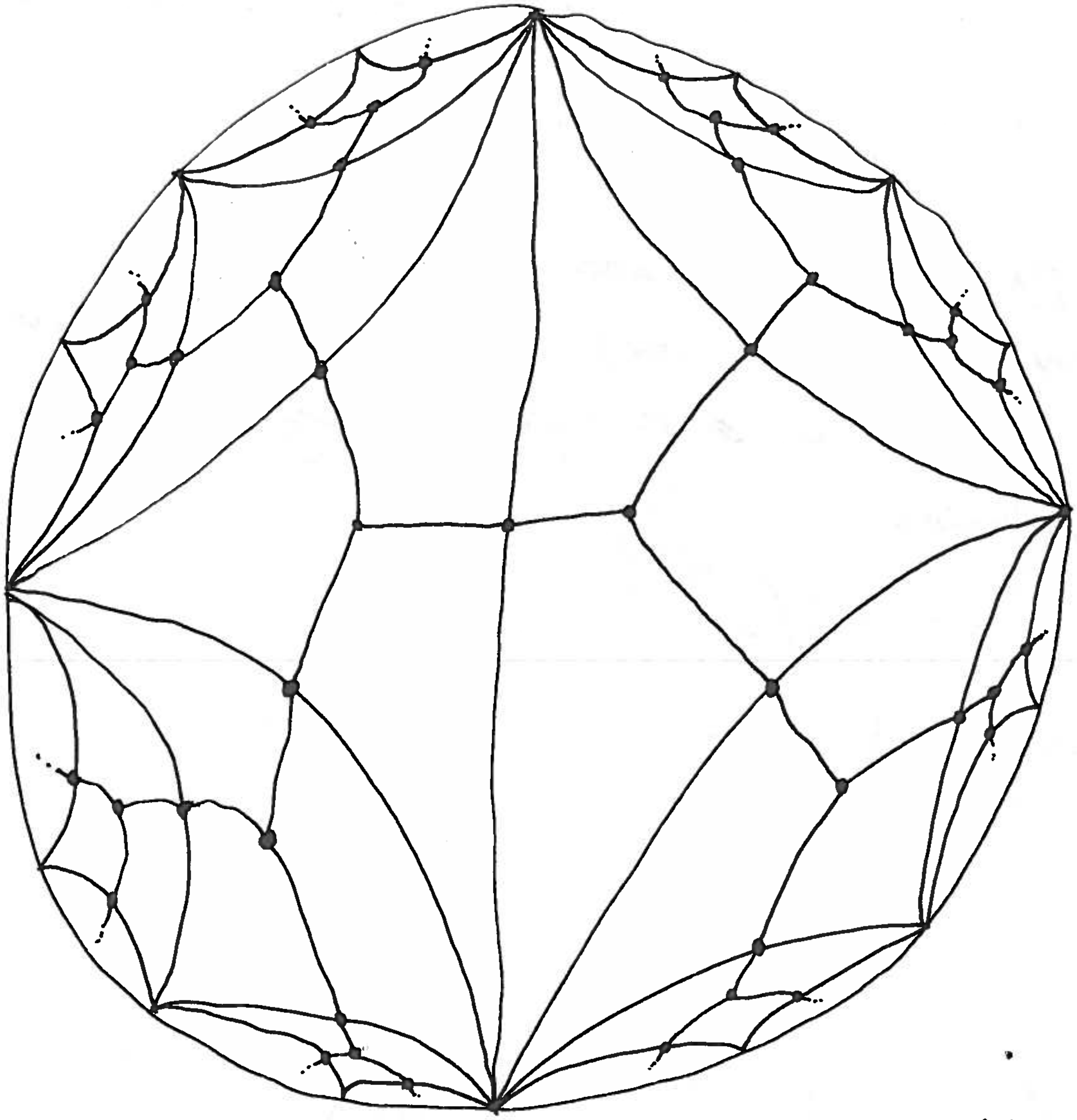
<u>e.g.</u>

B $\qquad\qquad$ B°

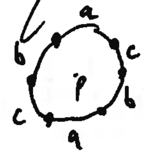
We are now ready to state the main result of this section.

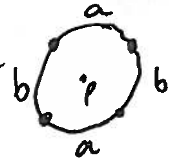
<u>Theorem (Harer)</u>: There is an explicit $\mathrm{Mod}_{g,1}$-equivariant deformation retraction $A^\circ \twoheadrightarrow Y^\circ$.

We will first see how this works in the $g=n=1$ case.



The trivalent graph dual to the Farey triangulation is Y. Depicted here is the first barycentric subdivision Y°.

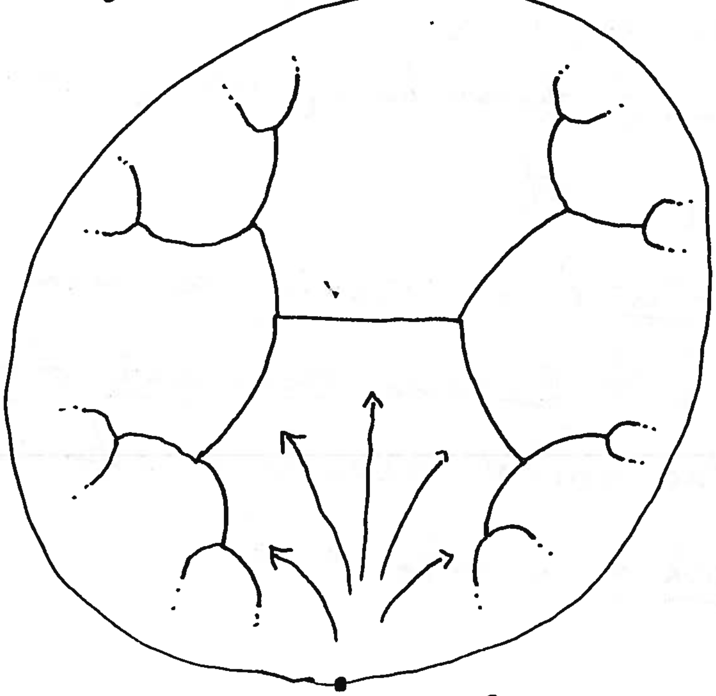
Notice that the trivalent vertices of Y° are realized in $T_{1,1}$ by Riemann surfaces whose horocyclic quadratic differential presents the surface as a regular hexagon 

$a=b=c$, while the bivalent vertices are realized in $T_{1,1}$ by Riemann surfaces whose horocyclic quadratic diff'l presents the surface as a square 

$a=b$.

Note also that such Riemann surfaces are precisely those with nontrivial stabilizer in $\mathrm{Mod}_{1,1} = PSL_2\mathbb{Z}$.

Re-drawing the picture in a less cluttered way,



$0 \in \partial\mathbb{H}^2 = \mathbb{R} \cup \{\infty\}$

we see that the indicated region meeting $\partial\mathbb{H}^2$ at 0 deformation retracts onto a portion of Y^0. We can use $\mathrm{Mod}_{1,1}$ to re-position any connected complementary region to Y^0 so that it is the indicated one, and hence *every* complementary region deformation retracts onto a portion of Y^0 in just this way. This completes the proof of the main theorem in the $g=n=1$ case.

In the general case of $T_{g,1}$, we proceed iteratively over the vertices of weight k of A°_∞.

The weight 0 vertices of A°_∞ correspond to arc systems made up of a single curve α_0. Consider the surface S obtained by deleting α_0. S has at most two connected components; denote them by S_1 and S_2, where S_2 is possibly empty, but S_1 is not.

Recall that the <u>star</u> of a vertex v is the union of simplices with v as a 0-face, and the <u>closure</u> of a set of simplices is the smallest subcomplex containing each of them. The <u>link</u> of a vertex v is:

$$\mathrm{Link}(v) = \mathrm{Closure}(\mathrm{Star}(v)) - \mathrm{Star}(v).$$

Finally, recall that the <u>join</u> $B * C$ of two simplicial complexes is the simplicial complex with a $(k+l+1)$-simplex for every pair (α,β) of a k-simplex α of B and an l-simplex β of C, where we allow $k=-1$ or $l=-1$ (a (-1)-simplex is the empty face $\emptyset$).

Then, for a weight 0 vertex v of A°_∞, we have

$$\mathrm{Link}(v) = A(S_1) * A(S_2).$$

This can be seen by thinking of $\mathrm{Link}(v)$ as the collection of weighted arc systems (α, w) where $\alpha_0 \in \alpha$, and $w_0 = 0$.

By a theorem of Harer,

Theorem (Harer):

(a) If E is a surface-with-boundary with marked points, and $E \setminus \{\text{marked points}\}$ is not a 2-disk or a once-punctured 2-disk, then the complex $A(E)$ is contractible.

(b) In the exceptional cases above, $A(E)$ is homeomorphic to a sphere.

Since v is a vertex of A^0_∞, we know S_1 and S_2 are not of the exceptional cases, and so $\mathrm{Link}(v)$ is contractible. We can thus retract A^0 off of the weight 0 vertices of A^0_∞.

Continuing iteratively with v a weight k vertex of A^0_∞, $\mathrm{Link}(v) = A(S_1) * \cdots * A(S_{k+2})$, and each S_i is not exceptional since $v \in A^0_\infty$. Thus A^0 retracts off all simplices with a face in A^0_∞, i.e. A^0 retracts onto Y^0.

$\square$

References:

[H86] Harer, J. The virtual cohomological dimension of the mapping class group of an orientable surface. Invent. math. 84, 157-176 (1986).

[H88] Harer, J. The cohomology of the moduli space of curves. In: Sernesi E. (eds) Theory of Moduli. Lecture Notes in Mathematics, vol 1337. Springer, Berlin, Heidelberg. (1988).

ii) Definition of A_∞

A family of curves is said to fill the surface F if each component of its complement is simply connected. Define A_∞ to be the subcomplex of A consisting of all simplices $<\alpha_0,\ldots,\alpha_k>$ such that $\{\alpha_i\}$ does not fill F. There is a natural action of Γ^1_g on A given by $[f]\cdot<\alpha_0,\ldots,\alpha_k> = <f(\alpha_0),\ldots,f(\alpha_k)>$ and since this action preserves A_∞ it restricts to an action on $A-A_\infty$. The main theorem of this chapter is:

Theorem 2.1: There is a homeomorphism $\omega\colon T^1_g \to A - A_\infty$ which commutes with the action of the mapping class group Γ^1_g.

Sections 2 and 3 of this chapter are devoted to the proof of this theorem. Before we go on, however, we will discuss in some detail the case where $g = 1$.

iii) Example, $g = 1$

A single simple closed curve cannot fill the torus, but any arc-system with 2 or more curves (rank = 1 or 2) must do so. This means that A_∞ contains only the vertices of A; these in turn may be identified with $\mathbb{Q} \cup \{\infty\}$: if $\{m,\ell\}$ is a basis for $\pi_1 F$ corresponding to two non-homotopic simple closed curves meeting only at $*$, then any other simple closed curve α through $*$ represents $a_1 m + a_2\ell$ in $\pi_1 F$ with a_1 prime to a_2. Associating a_1/a_2 to α gives the bijection between A_∞ and $\mathbb{Q} \cup \{\alpha\}$. It is easy to see that two curves with parameters (a_1,a_2) and (b_1,b_2) are isotopic to ones which meet only at $*$ exactly when $a_1b_2 - a_2b_1 = \pm 1$ We may therefore identify A with $\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$, where $\mathbb{H}$ is the hyperbolic plane as in figure 2.1 (upper half plane model) or figure 2.2 (Poincare model).

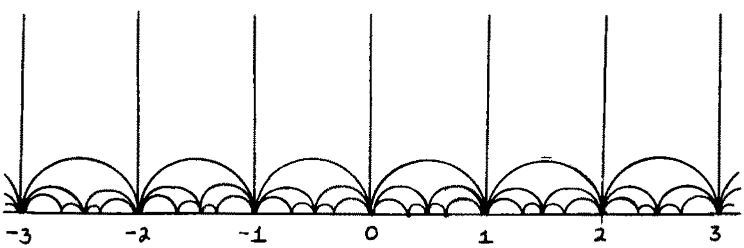


Figure 2.1

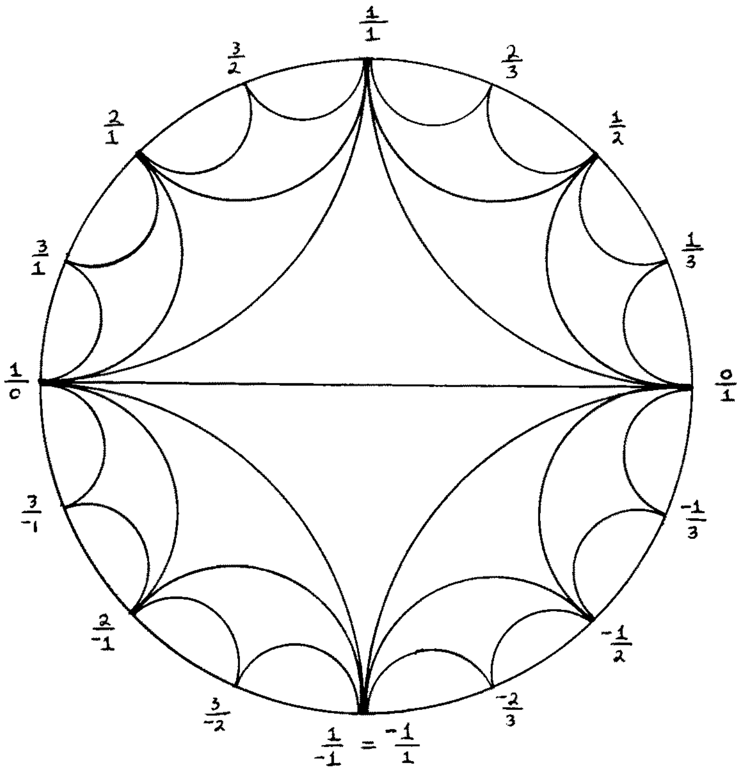


Figure 2.2

Since T_1^1 may be identified with $\mathbb{H}$, the picture gives an illustration of theorem 2.1.

It should be pointed out that the situation is much more complicated in higher genus. Since it takes $2g$ curves to fill the surface, A_∞ contains the $2g - 2$ skelton of A; however, it also contains pieces of the skeleta of A up to codimension 2. The existence of these higher dimensional cells will turn out to be a red herring, however, because we will see in chapter 4 that A_∞ has the homotopy type of a wedge of spheres of dimension $2g - 2$.

§2 The Conformal Point of View: Strebel Quadratic Differentials.

In this section we will give the Mumford-Strebel proof of Theorem 2.1. The main ingredient is the theory of quadratic differentials.